1.3/1.4A – Investigating Quadratic Functions in Standard Form: $y = a(x \pm h)^2 \pm k$

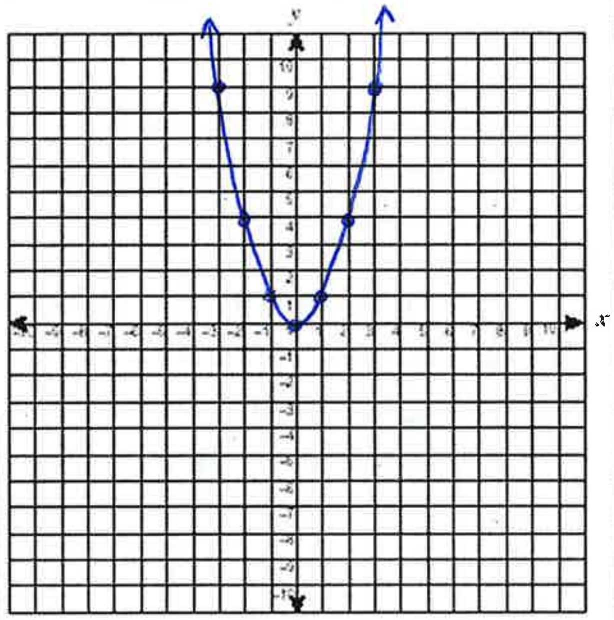
Key

Graph $y = x^2$ using a table of values

$y = x^2$

$y=(-3)^2=9$
$y=(-2)^2=4$
$y=(-1)^2=1$
$y=0^2=1$
$y=1^2=1$
etc.

x	y
-3	9
-2	4
-1	1
0	0
1	1
2	4
3	9



Quick way to graph:

Use a basic count:

Start at vertex: in this case (0,0)

Over 1, up 1
back to vertex
Over 2, up 4
back to vertex
Over 3, up 9
over 4, up 16
etc.

Graph Shape: the graph shape is called a parabola and occurs when the equation has an x^2 term as highest degree

Parabolas have a vertex, a middle point. For $y = x^2$, it is (0, 0)

Parabolas have an AXIS OF SYMMETRY, a reflection line that splits the parabola into two symmetrical branches. It can be shown with a dashed line.

In this example, the equation of the axis of symmetry is $x = 0$

Parabolas open upward or downward. If they open upwards, they go up forever and ever, but only go down so far. Therefore, they have a minimum value. In the example above, the minimum value is $y=0$. If they open downwards, they go down forever, but only go up so far. Therefore, they have a maximum value.

For any graph, you can find the domain.
How far left does the graph go? How far right?
In this example,
left forever, right forever so the domain is all values of x:

$x \in \mathbb{R}$ ← all real numbers
↑ belongs to

For any graph, you can find the range.
How far up does the graph go? How far down?
In this example,
up forever, down as low as $y=0$ (the minimum)

range is $y \geq 0$

A quadratic function is a function that has a second degree polynomial (has an x^2 term, but nothing higher. The graph shape that results is a PARABOLA.

degree

Examples: $y=x^2+1$, $y=x^2+2x-5$, $f(x) = 2(x-3)^2+7$

*Note: *f(x)* is the same as *y*

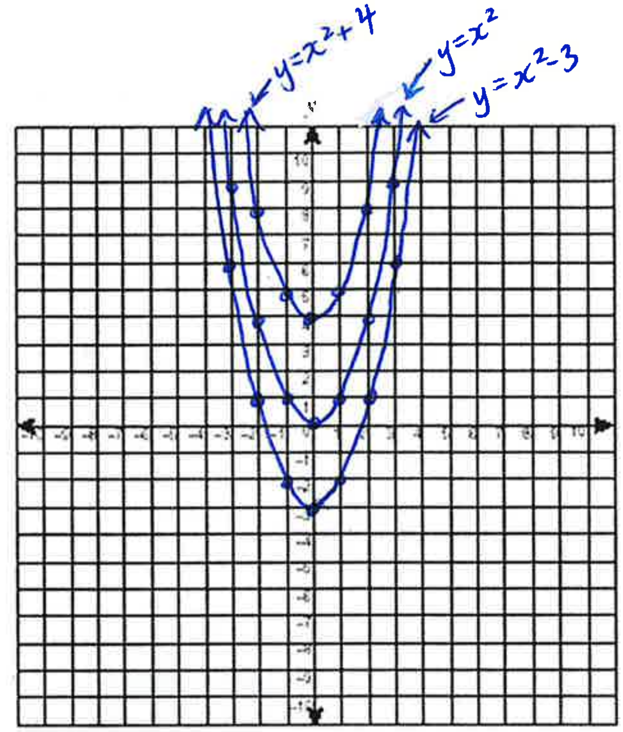
k value

$y = x^2 \pm k$

a) Graph $y = x^2$ using the basic count:
Start at (0,0) and go over 1, up 1

over 2, up 4

over 3, up 9



b) Graph $y = x^2 + 4$ using a table of values:

x	y
-3	13
-2	8
-1	5
0	4
1	5
2	8
3	13

$y=(-3)^2+4=9+4=13$
$y=(-2)^2+4=4+4=8$
$y=(-1)^2+4=5$
$y=0^2+4=4$
etc

Notice: $y=x^2+4$ is the same parabola as $y=x^2$, only translated (moved) up 4 units. This is due to the 'k' value of 4 in the function.

Vertex: $(0,4)$

A of S eqn: $x=0$

Max/Min: min @ $y=4$

Domain: $x \in \mathbb{R}$

Range: $y \geq 4$

c) Graph $y = x^2 - 3$ by count method:

k value is: -3

Vertex is: $(0,-3)$

Then do basic count: over 1, up 1; 2, 4; 3, 9

Vertex: $(0,-3)$ A of S eqn: $x=0$

Max/Min: min @ $y=-3$

Domain: $x \in \mathbb{R}$

Range: $y \geq -3$

$y = x^2 \pm k$

The *k* value: translates the parabola up (if k is positive) or down (if k is negative).

h value

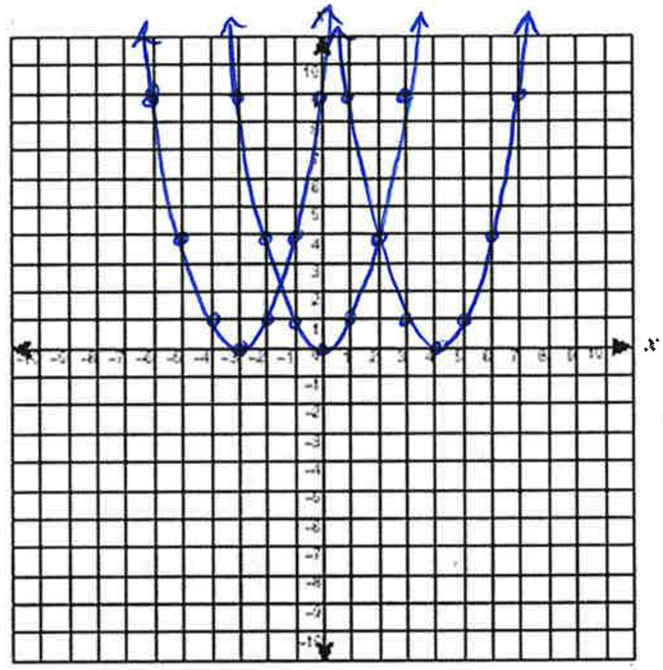
$y = (x \pm h)^2$

a) Graph $y = x^2$ using the count

b) Graph $y = (x-4)^2$ using a table of values

x	y
1	9
2	4
3	1
4	0
5	1
6	4
7	9

$y=(1-4)^2=(-3)^2=9$
$y=(2-4)^2=(-2)^2=4$
$y=(3-4)^2=(-1)^2=1$
$y=(4-4)^2=0^2=0$
etc...



Notice: $y=(x-4)^2$ is the same parabola as $y=x^2$ except it's translated 4 units right. This is due to the h value of 4.

Vertex: $(4,0)$

A of S eqn: $x=4$

Max/Min: min @ $y=0$

Domain: $x \in \mathbb{R}$

Range: $y \geq 0$

h value Mental Switch:
switch the sign of the constant in the brackets to get the 'h' value.

c) Graph $y = (x+3)^2$ using the count method:

Vertex: $(-3,0)$ Domain: $x \in \mathbb{R}$

A of S eqn: $x=-3$ Range: $y \geq 0$

Max/Min: min @ $y=0$

$y = (x \pm h)^2 \pm k$: vertical translation

horizontal translation
(sign switch necessary)

Vertex Notes: (h, k)

after sign switch

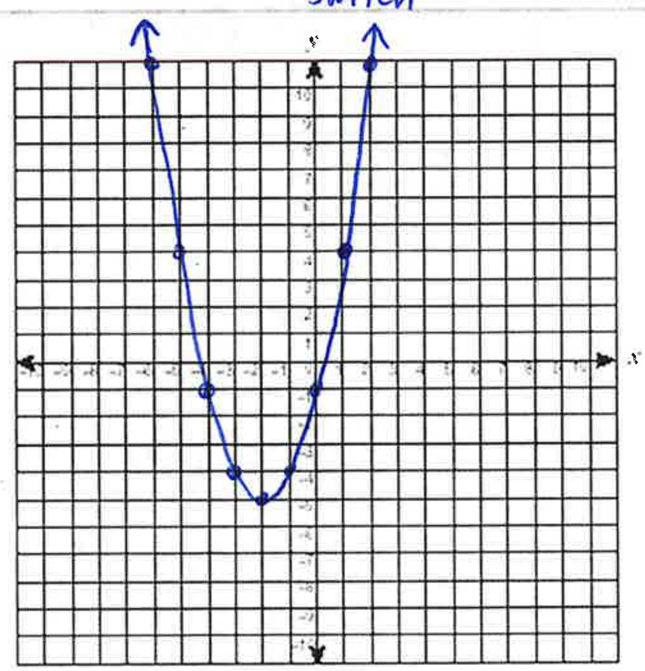
Practice

Example - Graph $y = (x+2)^2 - 5$ using the count method

vertex is $(-2,-5)$

from vertex, count

over	up
1	1
2	4
3	9
4	16



Vertex: $(-2,-5)$ Domain: $x \in \mathbb{R}$

A of S eqn: $x=-2$ Range: $y \geq -5$

Max/Min: min@ $y=-5$

1.4 B – Investigating Quadratic Functions in Standard Form: $y = a(x \pm h)^2 \pm k$

***a* value**

$y = ax^2$

a) Graph $y = x^2$ using the count.

b) Graph $y = 2x^2$ using a table of values

x	y
-3	18
-2	8
-1	2
0	0
1	2
2	8
3	18

$y = 2(-3)^2 = 2(9) = 18$

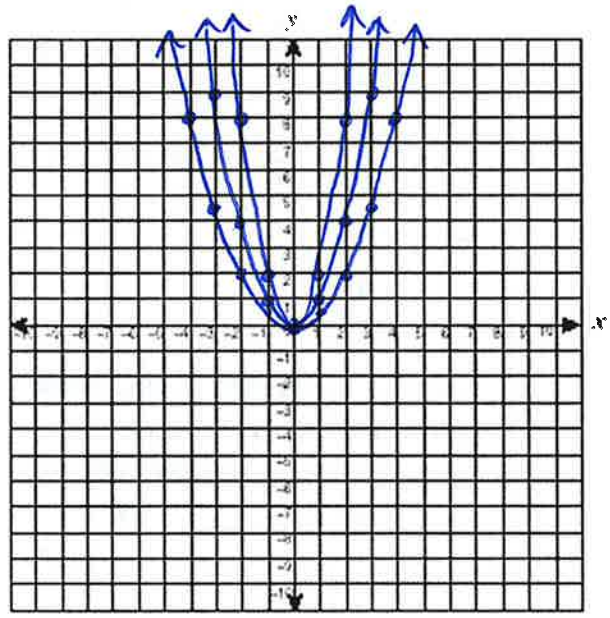
$y = 2(-2)^2 = 2(4) = 8$

$y = 2(-1)^2 = 2(1) = 2$

$y = 2(0)^2 = 2(0) = 0$

etc.

Notice: the parabola $y = 2x^2$ is skinnier (it rises faster) compared to $y = x^2$. This is due to the 'a' value of 2, which doubles the y value compared to the basic parabola $y = x^2$



The *a* value: alters the 'up' count
- if $a > 1$, parabola is skinnier
- if $a < 1$, parabola is wider

c) Graph $y = \frac{1}{2}x^2$ using the count method:

vertex $(0,0)$ over 1, up $\frac{1}{2}$ (half of 1)
2, up 2 (half of 4)
3, up 4.5
4, up 8

Graph $y = -x^2$ using a table of values

x	y
-3	-9
-2	-4
-1	-1
0	0
1	-1
2	-4
3	-9

$y = -(-3)^2 = -9$

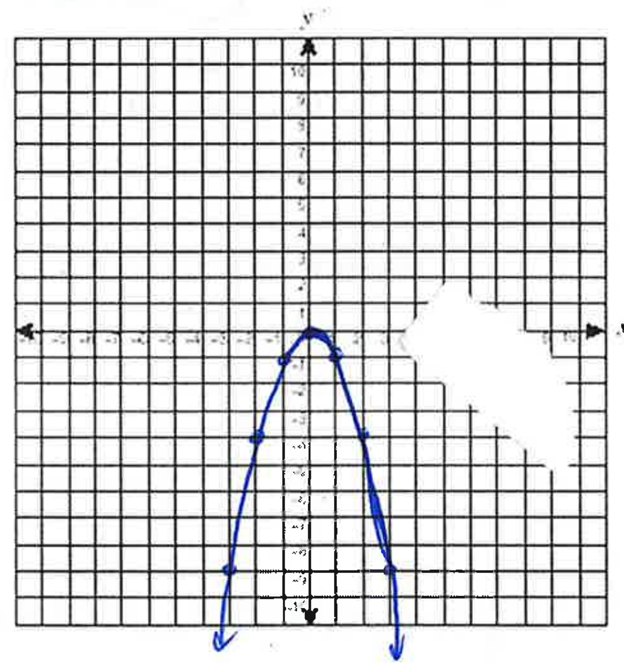
$y = -(-2)^2 = -4$

etc..

Vertex: $(0,0)$ Domain: $x \in \mathbb{R}$

A of S eqn: $x = 0$ Range: $y \leq 0$

Max/Min: Max @ $y = 0$



The $-a$ value: parabola opens down so all 'up' counts become 'down' counts

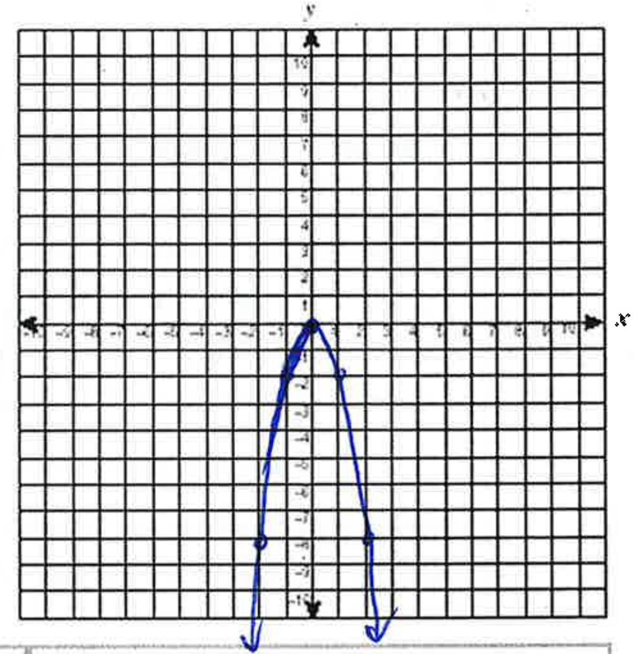
Graph $y=-2x^2$ using the count method

vertex $(0,0)$

over 1, down 2 (double 1)

over 2, down 8 (double 4)

over 3, down 18



Standard Form:

$y = \pm a(x \pm h)^2 \pm k$

alters up count (+) or down count (-)

horizontal translation (sign switch)

vertical translation

Notes: Vertex (h, k) after sign switch

A of S: $x = h$

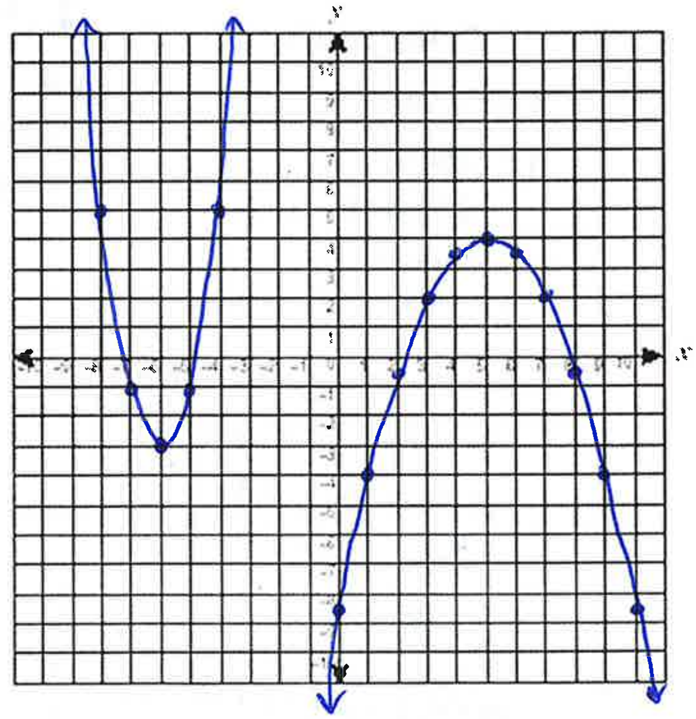
max/min: $y = k$

Graph a) $f(x) = 2(x+6)^2 - 3$

and b) $y - \cancel{4} = -\frac{1}{2}(x-5)^2 + 4$
$\cancel{+4}$

For each, find the

- vertex
- axis of sym eqn
- max/min
- domain
- range



a) Vertex: $(-6,-3)$

A of S eqn: $x=-6$

Max/Min: min @ $y=-3$

Domain: $x \in \mathbb{R}$

Range: $y \geq -3$

b) Vertex: $(5,4)$

A of S eqn: $x=5$

Max/Min: max @ $y=4$

Domain: $x \in \mathbb{R}$

Range: $y \leq 4$

(a) over 1, up 2
over 2, up 8
3, 18

(b) over 1, down $\frac{1}{2}$

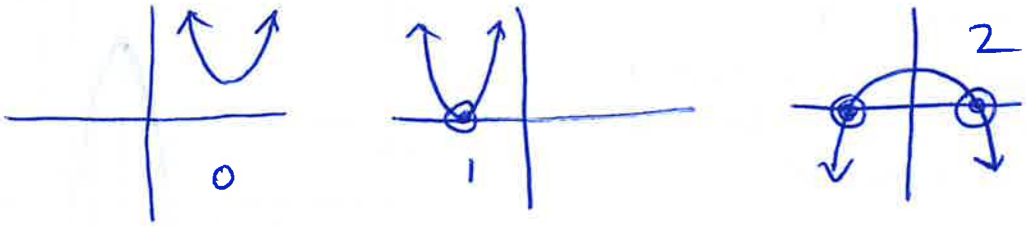
2	2
3	4.5
4	8
5	12.5

x-ints

Thinking back to last chapter, what are *x*-intercepts?

where the graph touches or crosses the x axis.

How many *x*-intercepts for a quadratic function? $0, 1,$ or 2



What are the methods we learned to identify *x*-intercepts?

set y to 0 and solve by factoring, complete the square, or quadratic formula

Example 1 – Determine the number of x-intercepts for each quadratic function, and also determine the *y*-intercept of each.

a) $y = -2(x-7)^2 - 1$

vertex $(7, -1)$
- below x axis

$a = -2$ so it opens down

$\therefore$ NO x-ints

y-int; set $x=0$

$y = -2(0-7)^2 - 1$
$= -2(49) - 1$
$= -98 - 1$
$= -99$

b) $y = 0.5x^2 - 6$

vertex $(0, -6)$
below x axis
$a = 0.5$ so opens up

$\therefore$ TWO x-ints

y-int:
$y = 0.5(0)^2 - 6$
$y = -6$

c) $y = -2(x+1)^2$

vertex $(-1, 0)$
is ON the x-axis
so

1 x-int

y-int:
$y = -2(0+1)^2$
$y = -2$

Example 2 – Write a quadratic function with a maximum of 3, axis of symmetry equation *x= -1*, that passes through (1, 1).

a value is neg

vertex $(-1, 3)$

$y = a(x+1)^2 + 3$

passes through $(\underset{x}{1}, \underset{y}{1})$

$1 = a(1+1)^2 + 3$

$1 = a(2)^2 + 3$

$1 = 4a + 3$
$-3 \qquad -3$

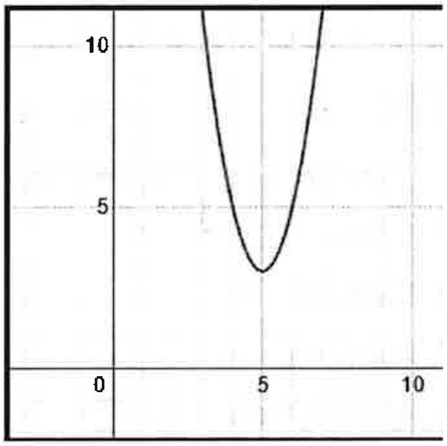
$-2 = 4a$

$a = -\frac{1}{2}$

$y = -\frac{1}{2}(x+1)^2 + 3$

2.1 – Finding the Equation of a Parabola

Example 1 – Determine the equation of the following parabola:



vertex $(5, 3)$

over 1, up 2

over 2, up 8

so $a = 2$

$$y = 2(x-5)^2 + 3$$

Example 2 – Find the equation of a quadratic function whose graph has vertex (4, 8) and an *x*-intercept of 6.

$$y = a(x-4)^2 + 8$$

x-int of 6 means the parabola passes through $(6, 0)$ (x, y)

$$0 = a(6-4)^2 + 8$$

$$0 = a(2)^2 + 8$$

$$0 = 4a + 8$$

$$-8 = 4a$$

$$-2 = a$$

$$y = -2(x-4)^2 + 8$$

Example 3 – A parabola with vertex (1, -2) passes through the point (4, 1). Find the equation.

$$y = a(x-1)^2 - 2$$

$(4, 1)$ (x, y)

$$1 = a(4-1)^2 - 2$$

$$1 = a(3)^2 - 2$$

$$1 = 9a - 2$$

$$3 = 9a$$

$$\frac{1}{3} = a$$

$$y = \frac{1}{3}(x-1)^2 - 2$$

Example 4 – Find an equation of a quadratic function with points (3, -4), (-3, 2), & (1, 2).

$(-3, 2)$ and $(1, 2)$ are symmetric due to same y value. So the axis of symmetry must be midway between -3 and 1, so -1

so $h = -1$

so $y = a(x+1)^2 + k$

sub in a point, say $(1, 2)$

$$2 = a(1+1)^2 + k$$

$$2 = 4a + k$$

$$k = 2 - 4a$$

So now $y = a(x+1)^2 + 2 - 4a$

now sub in another point, say $(3, -4)$

$$-4 = a(3+1)^2 + 2 - 4a$$

$$-4 = 16a + 2 - 4a$$

$$-6 = 12a$$

$$a = -\frac{1}{2} \Longrightarrow k = 2 - 4a = 2 - 4(-\tfrac{1}{2}) = 2 + 2 = 4$$

$$y = -\frac{1}{2}(x+1)^2 + 4$$

2.2 – Completing the Square

completing the square

When quadratic functions are in GENERAL FORM [$y = ax^2 \pm bx \pm c$], they can be changed into STANDARD FORM [$y = (x \pm p)^2 \pm q$] using a technique called completing the square.

Example 1 - Rewrite $y = 14 + 10x + x^2$ in standard form by completing the square. Then sketch the graph. Calculate the x-intercepts.

STEPS:

1) Rearrange so squared term is first and x term is second.
2) Find the a, b, c values
3) Take half the b-value (you'll need this later), then square it.
4) Add and subtract the result to your quadratic function after the x term.
5) Make sure the new term you added is the third term.
6) Factor the trinomial and add the two last terms.

Shortcut for factoring the trinomial:

When searching for two numbers that multiply to c and add to b, the two numbers will always be the halved 'b' value

① $y = x^2 + 10x + 14$

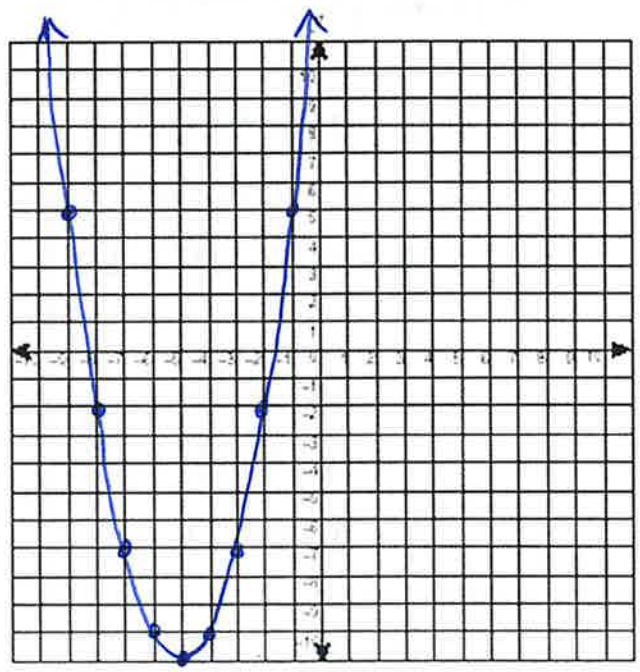
② $a = 1,\ b = 10,\ c = 14$

③ $b = 10,\ \frac{10}{2} = 5,\ 5^2 = 25$

④⑤ $y = x^2 + 10x + 25 - 25 + 14$

⑥ $y = (x^2 + 10x + 25) - 25 + 14$

$y = (x+5)^2 - 11$



Example 2- Change $y = x^2 - 4x - 1$ into standard form, then calculate the x-intercepts.

$y = x^2 - 4x - 1$

$a = 1,\ b = -4,\ c = -1$

$-4;\ -\frac{4}{2} = -2,\ (-2)^2 = 4$

$y = x^2 - 4x + 4 - 4 - 1$

$y = (x^2 - 4x + 4) - 4 - 1$

$y = (x-2)^2 - 5$

x-ints: set $y = 0$

$0 = (x-2)^2 - 5$

$\pm\sqrt{5} = \sqrt{(x-2)^2}$

$\pm\sqrt{5} = x - 2$

$x = 2 \pm \sqrt{5}$

When a $\neq$ 1

When the *a* value is different from 1, there are a few more steps.

Example 3 - Change $y = -2x^2 + 4x + 5$ into standard form and then find *x*-ints

STEPS:

1) Group the first two terms together.
2) Factor the *a* value out.
3) Find the *b* value. Take half and square it.
4) Add and subtract the result IN THE BRACKETS.
5) Get the subtracted result out of the brackets by multiplying to the coefficient in front of the brackets.
6) Factor the trinomial.

① $y=(-2x^2+4x)+5$

② $y=-2(x^2-2x)+5$

③ $b=-2;\ -1;\ 1$

④ $y=-2(x^2-2x+1-1)+5$

⑤ $y=-2(x^2-2x+1-1)+5$

$y=-2(x^2-2x+1)+2+5$

⑥ $y=-2(x-1)^2+7$

x-ints:

$0=-2(x-1)^2+7$

$\frac{7}{2}=(x-1)^2$

$\pm\sqrt{\frac{7}{2}}=x-1$

$1\pm\sqrt{\frac{7}{2}\cdot\frac{\sqrt{2}}{\sqrt{2}}}=x$

$1\pm\frac{\sqrt{14}}{2}=x$

$\frac{2\pm\sqrt{14}}{2}=x$

Example 4 - Change $y = 3x^2 - 12x + 11$ into vertex form, then calculate the *x*-ints.

$y=(3x^2-12x)+11$

$y=3(x^2-4x)+11$

$-4;\ -2;\ 4$

$y=3(x^2-4x+4-4)+11$

$y=3(x^2-4x+4)-12+11$

$y=3(x-2)^2-1$

$0=3(x-2)^2-1$

$\frac{1}{3}=(x-2)^2$

$\pm\sqrt{\frac{1}{3}}=x-2$

$2\pm\sqrt{\frac{1}{3}\cdot\frac{\sqrt{3}}{\sqrt{3}}}=x$

$\frac{2^{\times 3}}{1_{\times 3}}\pm\frac{\sqrt{3}}{3}=x$

$x=\frac{6\pm\sqrt{3}}{3}$

Example 5 - Change $y = 5x - 3x^2 + 1$ into standard form. Do this both in 'fraction form' and 'decimal form'.

$y=(-3x^2+5x)+1$

$y=-3(x^2-\frac{5}{3}x)+1$

$b=-\frac{5}{3};\ -\frac{5}{6};\ \frac{25}{36}$

$y=-3(x^2-\frac{5}{3}x+\frac{25}{36}-\frac{25}{36})+1$

$y=-3(x^2-\frac{5}{3}x+\frac{25}{36})+\frac{75}{36}+\frac{1^{\times 36}}{1_{\times 36}}$

$y=-3(x-\frac{5}{6})^2+\frac{111^{\div 3}}{36_{\div 3}}$

$y=-3(x-\frac{5}{6})^2+\frac{37}{12}$

$y=(-3x^2+5x)+1$

$y=-3(x^2-1.667x)+1$

$b=-1.667;\ -0.8\overline{3},\ 0.694$

$y=-3(x^2-1.667x+0.694-0.694)+1$

$y=-3(x^2-1.667x+0.694)+2.083+1$

$y=-3(x-0.8\overline{3})^2+3.083$

2.4 – Applications of Quadratic Functions

Example 1 - The path of a rocket fired over a lake is described by the function

$h(t) = -4.9t^2 + 49t + 1.5$ where *h(t)* is the height of the rocket, in metres, and *t* is time in seconds, since the rocket was fired.

a) What is the maximum height reached by the rocket? How many seconds after it was fired did the rocket reach this height?
b) How high was the rocket above the lake when it was fired?
c) At what time does the rocket hit the ground?
d) What domain and range are appropriate in this situation?
e) How high was the rocket after 7s? Was it on its way up or down?

(a) Max height occurs at the vertex, so function must be changed to standard form

$h(t) = (-4.9t^2 + 49t) + 1.5$

$h(t) = -4.9(t^2 - 10t) + 1.5$

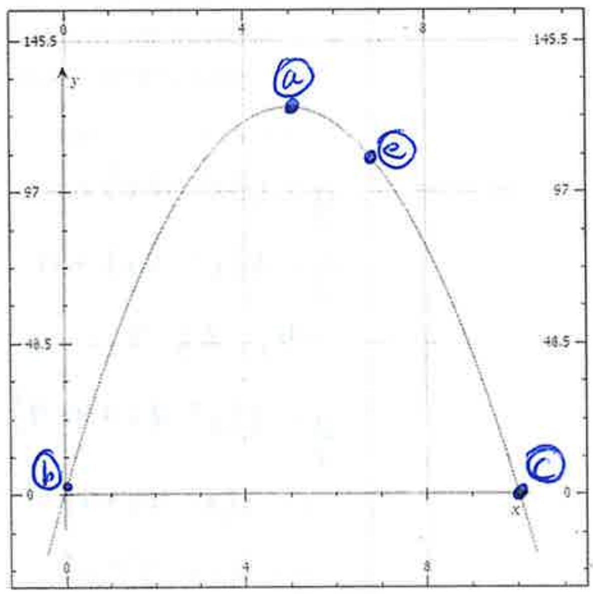
$h(t) = -4.9(t^2 - 10t + 25 - 25) + 1.5$

$h(t) = -4.9(t^2 - 10t + 25) + 122.5 + 1.5$

$h(t) = -4.9(t-5)^2 + 124$

vertex $(5, 124)$

The max height reached by the rocket is 124m, five seconds into its flight.



(b) $t=0$ when rocket fired.

$h(0) = -4.9(0-5)^2 + 124$

$h(0) = -4.9(-5)^2 + 124 = -4.9(25) + 124$

$h(0) = -122.5 + 124$

$h(0) = 1.5m$ The rocket was 1.5m above the ground when fired

(c) $h=0$ when rocket hits ground

$0 = -4.9(t-5)^2 + 124$

$25.306 = (t-5)^2$

$5.03 = t - 5$

$t = 10.03s$

The rocket hits the ground after 10.03s.

(d)

Domain: $0 \le t \le 10.03s$ or $\{t \mid 0 \le t \le 10.03\}$

Range: $0 \le h \le 124m$ or $\{h \mid 0 \le h \le 124\}$

(e) $h(7) = -4.9(7-5)^2 + 124$

$h(7) = -4.9(2)^2 + 124$

$h(7) = -19.6 + 124$

$h(7) = 104.4m$

On its way down

***Keep in mind that the question presented this function in general form. Sometimes, in problems like this, the function is presented in standard form, which will make it much easier**

Example 2 – At a concert, organizers are roping off a rectangular area for sound equipment. There is 160m of fencing available to create the perimeter. What dimensions will give the maximum area, and what is the maximum area?

Steps:

1) Write an equation for perimeter, and write an equation for area for a rectangle.
2) Use the two equations to create a quadratic function in general form.
3) Complete the square to change the quadratic function into standard form.
4) Identify the maximum area, and then the dimensions for the maximum area.

Let $x =$ length and $y =$ width

(1) $160 = 2x + 2y$

$A = xy$

(2) $\frac{2y}{2} = \frac{160}{2} - \frac{2x}{2}$

$y = 80 - x$

$A = x(80 - x)$

$A = 80x - x^2$

$A = -x^2 + 80x$

(3) $A = (-x^2 + 80x)$

$A = -1(x^2 - 80x)$

$A = -1(x^2 - 80x + 1600 - 1600)$

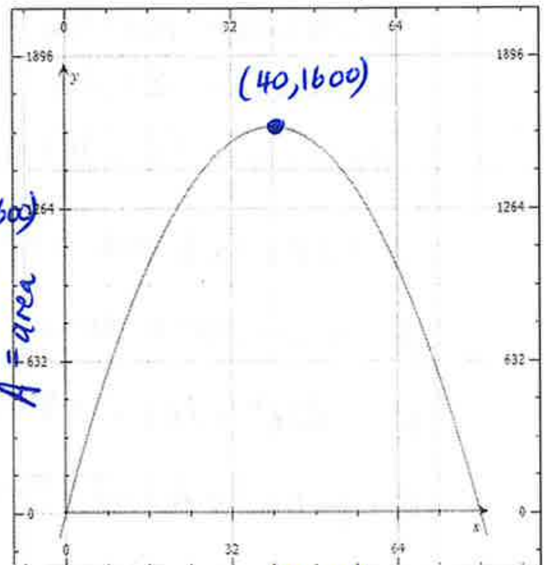
$A = -1(x^2 - 80x + 1600) + 1600$

$A = -(x - 40)^2 + 1600$

Vertex $(\underset{x}{40}, \underset{A}{1600})$

Vertex occurs at max of parabola

Area



$x =$ length

(4) The maximum area is $1600m^2$

The length of the rectangle that allows for max area is 40m

The width is: $2(40) + 2y = 160$

$80 + 2y = 160$

$2y = 80$

$y = 40$

40m so 40m x 40m

Example 3 – A rancher has 800m of fencing to enclose a rectangular cattle pen along a river bank. There is no fencing needed along the river bank. Find the dimensions that would enclose the largest area. Let $x =$ length

Let $y =$ width.

x y x

$2x + y = 800$

$A = xy$

$y = 800 - 2x$

$A = x(800 - 2x)$

$A = 800x - 2x^2$

$A = -2x^2 + 800x$

$A = -2(x^2 - 400x)$

$A = -2(x^2 - 400x + 40000 - 40000)$

$A = -2(x^2 - 400x + 40\,000) + 80\,000$

$A = -2(x - 200)^2 + 80\,000$

vertex $(\underset{x}{200}, \underset{A}{80\,000})$

$2x + y = 800$

$2(200) + y = 800$

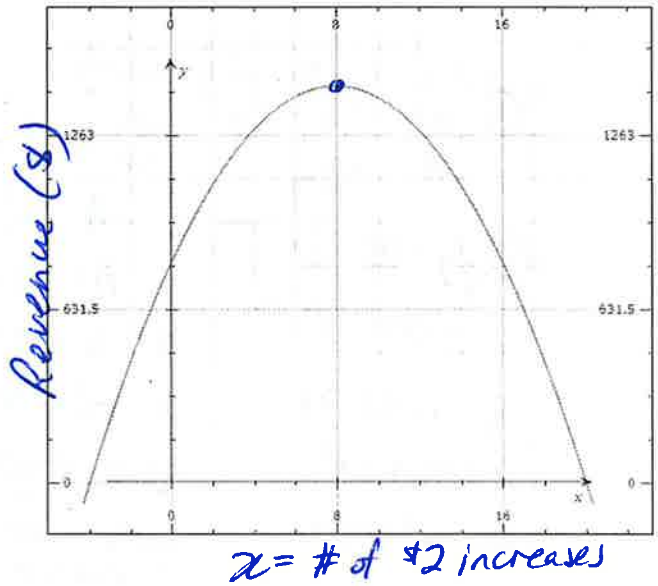
$400 + y = 800$

$y = 400$

The dimensions that give the max area of $80000m^2$ are 400m x 200m

Example 4 – A sporting goods store sells basketball shorts for \$8. At this price their weekly sales are approximately 100 items. Research says that for every \$2 increase in price, the manager can expect the store to sell five fewer pairs of shorts. Determine the maximum revenue the manager can expect based on these estimates. What selling price will give that maximum revenue, and how many shorts will be sold?

Let x = # of \$2 increases in price



Shorts sold = $100-5x$

Price = $8+2x$

Revenue = (Price)(# sold)

$R=(8+2x)(100-5x)$

$R=800+200x-40x-10x^2$

$R=-10x^2+160x+800$

$R=(-10x^2+160x)+800$

$R=-10(x^2-16x)+800$

$R=-10(x^2-16x+64-64)+800$

$R=-10(x^2-16x+64)+640+800$

$R=-10(x-8)^2+1440$

Vertex $(\underset{x}{8}, \underset{R}{1440})$

Price = $8+2x=8+2(8)=\$24$

shorts sold = $100-5x=100-5(8)=60$

To maximize revenue (\$1440), the shorts should be sold for \$24 ea, and 60 pairs will be sold per week.