

3.1 – Solving Quadratic Equations by Factoring

Key

A function of *degree 2* (meaning the highest exponent on the variable is 2), is called a **Quadratic Function**.

Quadratic functions are written as, for example, $f(x) = x^2 - x - 6$ OR $y = x^2 - x - 6$. $f(x)$ is 'f of x' and means that the y value is dependent upon the value of x. Once you have an x value and you substitute it into the function, the value of $f(x)$ will result. $f(x)$ is really just another way of writing y, and is used when the graph is a function (passes the vertical line test).

When a quadratic function is graphed, a **parabola** results.

Example 1: $y = x^2$

$$y = (-3)^2 = 9$$

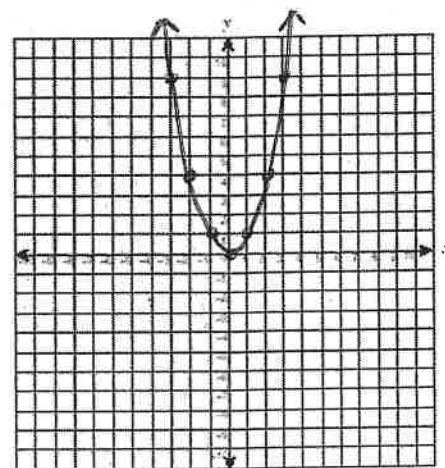
$$y = (-2)^2 = 4$$

$$y = (-1)^2 = 1$$

$$y = 0^2 = 0$$

etc

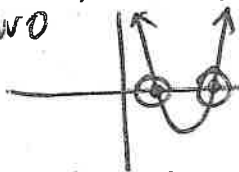
x	y
-3	9
-2	4
-1	1
0	0
1	1
2	4
3	9



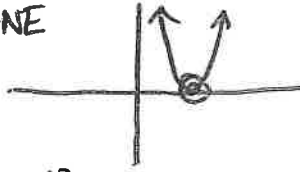
A very important feature of the parabola that results from a quadratic function is where it touches or crosses the x-axis. These are the **x-intercepts** of the parabola.

How many **x-intercepts** can a parabola have? Draw all possibilities:

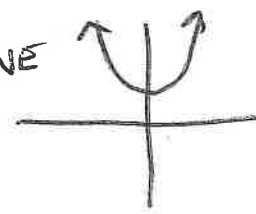
TWO



ONE



NONE



What is the y value at an x-intercept? $y = 0$ at an x-int

Therefore, to find the x-intercepts of a parabola, we can set $y = 0$ (or $f(x) = 0$) and solve the resulting **quadratic equation**.

When y is set to 0, we call the question a **quadratic equation** instead of a quadratic function.

Quadratic function: $f(x) = x^2 - x - 6$ OR $y = x^2 - x - 6$

Quadratic equation: $x^2 - x - 6 = 0$

The **x-intercepts** of the parabola are the **zeros** of the quadratic function. They are also called the **solutions** or **roots** of the quadratic equation.

Therefore, how many **zeros** can there be for a quadratic function? 0, 1, or 2

How many **roots** or **solutions** can there be for a quadratic equation? 0, 1, or 2

One method to find the **zeros** of a **quadratic function** is to graph it and visually determine the **x-intercepts**.

You can often find the roots of a quadratic equation by factoring when in general form $ax^2 + bx + c = 0$. Remember, the roots or solutions of the quadratic equation correspond to the zeros of the quadratic function, and the x-intercepts of the parabola.

if $a = 1$

Example 2 - Solve and check $x^2 + 3x = -2$

To 'solve' a quadratic equation means to find the roots or solutions. Steps are as follows:

- 1) Get everything to one side so that only zero is on the other.
- 2) Identify a , b , and c values, and factor accordingly into binomials.
- 3) The roots are the x -values that will make the product of the binomials zero. If either of the binomials equal zero, then the product of the binomials will equal zero (this is called the Zero Factor Property). Therefore, identify the x values that make each binomial equal to zero.

Solve and check $x^2 + 3x = -2$

① $x^2 + 3x + 2 = 0$

$\times 2, +3$

② $a=1, b=3, c=2$

(2, 1)

Find two numbers that multiply to c and add to b

$(x+2)(x+1) = 0$

If either bracket equals 0, the eqn equals 0
 ③ $(x+2)(x+1) = 0$
 \uparrow \uparrow
 -2 makes this bracket zero -1 makes this bracket zero
 $x = -2, -1$

Check: $\underline{-2}$

$x^2 + 3x = -2$

$(-2)^2 + 3(-2) = -2$

$4 + -6 = -2$

✓

$\underline{-1}$

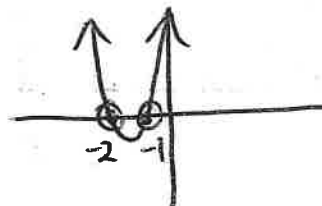
$x^2 + 3x = -2$

$(-1)^2 + 3(-1) = -2$

$1 + -3 = -2$

✓

Sketch the Graph:



Example 3: a) Solve $x^2 - 8x - 40 = 8$

$x^2 - 8x - 48 = 0$

$(x-12)(x+4) = 0$

$x = 12, -4$

b) Solve $2x^2 + 6x - 108 = 0$

$2(x^2 + 3x - 54) = 0$

$2(x+9)(x-6) = 0$

$x = -9, 6$

c) Solve $\frac{1}{2}x^2 - x - 4 = 0$

$\frac{1}{2}(x^2 - 2x - 8) = 0$

$\frac{1}{2}(x-4)(x+2) = 0$

$x = 4, -2$

d) Solve $\frac{x}{x-5} - \frac{3}{x+1} = \frac{30}{x^2 - 4x - 5}$

$\frac{x}{x-5} - \frac{3}{x+1} = \frac{30}{(x-5)(x+1)}$

$x \neq 5, -1$

$(x-5)(x+3)$

$x(x+1) - 3(x-5) = 30$

$x^2 + x - 3x + 15 = 30$

$x^2 - 2x - 15 = 0$

$x = -3$

$x = -3$

if $a \neq 1$

When $a \neq 1$, factor by "decomposition."

Example 4 - Solve $3x^2 - 5x + 2 = 0$

$$a=3, b=-5, c=2 \quad \begin{array}{r} x \quad ac \quad + \quad b \\ \times 6 \quad + \quad -5 \\ \hline \quad \quad -3, -2 \end{array}$$

$$\begin{array}{l} 3x^2 - 3x - 2x + 2 = 0 \\ 3x(x-1) - 2(x-1) = 0 \\ (x-1)(3x-2) = 0 \end{array} \quad \left| \begin{array}{l} x-1=0 \\ x=1 \end{array} \right. \quad \begin{array}{l} 3x-2=0 \\ 3x=2 \\ x=\frac{2}{3} \end{array}$$

$x = 1, \frac{2}{3}$

Trick for finding roots:

$$(3x-2) = 0$$

'Right over left, switch sign'

$$x = \frac{2}{3}$$

Example 5 - a) Solve $3(x^2 - 2) = -7x$ b) Solve $2(3x+2)^2 - 5(3x+2) = -2$

$$\begin{array}{l} 3x^2 - 6 = -7x \\ 3x^2 + 7x - 6 = 0 \\ 3x^2 + 9x - 2x - 6 = 0 \\ 3x(x+3) - 2(x+3) = 0 \\ (x+3)(3x-2) = 0 \\ x = -3, \frac{2}{3} \end{array}$$

Let $m = (3x+2)$

$$\begin{array}{l} 2m^2 - 5m = -2 \\ 2m^2 - 5m + 2 = 0 \\ 2m^2 - 4m - m + 2 = 0 \\ 2m(m-2) - 1(m-2) = 0 \\ (m-2)(2m-1) = 0 \\ m = 2, \frac{1}{2} \end{array}$$

$$3x+2=2$$

$$3x=0$$

$$x=0$$

$$3x+2 = \frac{1}{2}$$

$$3x = -\frac{3}{2}$$

$$x = -\frac{3}{6} = -\frac{1}{2}$$

$$x = 0, -\frac{1}{2}$$

If $c = 0$

Example 6 - Solve $3x^2 = -5x$

$$\begin{array}{l} 3x^2 + 5x = 0 \\ x(3x+5) = 0 \\ x = 0, -\frac{5}{3} \end{array}$$

difference of squares

Example 7 - a) Solve $x^2 - 25 = 0$

$$\begin{array}{l} (x+5)(x-5) = 0 \\ x = \pm 5 \end{array}$$

b) Solve $8p^2 - 18 = 0$

$$\begin{array}{l} 2(4p^2 - 9) = 0 \\ 2(2p+3)(2p-3) = 0 \\ p = \pm \frac{3}{2} \end{array}$$

c) Solve $49 - 4x^2 = 0$

$$(7+2x)(7-2x) = 0$$

can't use 'trick' as x term in second position.

$$7+2x=0 \quad 7-2x=0$$

$$2x=-7 \quad 7=2x$$

$$x = -\frac{7}{2} \quad x = \frac{7}{2}$$

Example 8 - a) Write a quadratic equation with roots 6 and -1. b) $\frac{2}{3}$ and $-\frac{1}{2}$

$$\textcircled{a} \quad \begin{array}{l} (x-6)(x+1) = 0 \\ x^2 - 5x - 6 = 0 \end{array} \quad \textcircled{b} \quad \begin{array}{l} (3x-2)(2x+1) = 0 \\ 6x^2 - x - 2 = 0 \end{array}$$

Pre 3.2 – Simplifying Radicals

When a radical is simplified, an *entire radical* is changed to a *mixed radical* (or a mixed radical is further simplified). Example 1 – Simplify

$$\begin{aligned} \text{a) } \sqrt{18} &= \sqrt{9 \cdot 2} \\ &= \sqrt{9} \cdot \sqrt{2} \\ &= 3 \cdot \sqrt{2} \\ &= 3\sqrt{2} \end{aligned}$$

$$\begin{aligned} \text{b) } \sqrt{24} &= \sqrt{4 \cdot 6} \\ &= \sqrt{4} \cdot \sqrt{6} \\ &= 2\sqrt{6} \end{aligned}$$

$$\begin{aligned} \text{c) } 3\sqrt{50} &= 3 \cdot \sqrt{25 \cdot 2} \\ &= 3 \cdot \sqrt{25} \cdot \sqrt{2} \\ &= 3 \cdot 5 \cdot \sqrt{2} \\ &= 15\sqrt{2} \end{aligned}$$

$$\begin{aligned} \text{d) } \sqrt{72} &= \sqrt{36 \cdot 2} \\ &= \sqrt{36} \cdot \sqrt{2} \\ &= 6\sqrt{2} \end{aligned}$$

or

$$\begin{aligned} &= \sqrt{9 \cdot 8} \\ &= \sqrt{9} \cdot \sqrt{8} \\ &= 3\sqrt{8} \\ &= 3 \cdot \sqrt{4 \cdot 2} \\ &= 3 \cdot \sqrt{4} \cdot \sqrt{2} \\ &= 3 \cdot 2 \cdot \sqrt{2} \\ &= 6\sqrt{2} \end{aligned}$$

$$\begin{aligned} \text{e) } 2\sqrt{45} &= 2 \cdot \sqrt{9 \cdot 5} \\ &= 2 \cdot \sqrt{9} \cdot \sqrt{5} \\ &= 2 \cdot 3 \cdot \sqrt{5} \\ &= 6\sqrt{5} \end{aligned}$$

$$\begin{aligned} \text{f) } \sqrt{50} &= \sqrt{25 \cdot 2} \\ &= \sqrt{25} \cdot \sqrt{2} \\ &= 5\sqrt{2} \end{aligned}$$

A final answer cannot have a radical in the denominator. Therefore, you may have to '**rationalize the denominator**' – a process that will eliminate the radical from the denominator without changing the value of the expression.

Example 2 - Simplify: a) $\frac{3 \cdot \sqrt{11}}{\sqrt{11} \cdot \sqrt{11}}$

$$\frac{3\sqrt{11}}{11}$$

b) $\sqrt{\frac{2}{5}}$

$$\begin{aligned} &= \frac{\sqrt{2 \cdot 5}}{\sqrt{5 \cdot 5}} \\ &= \frac{\sqrt{10}}{5} \end{aligned}$$

c) $\frac{7 \cdot \sqrt{3}}{2\sqrt{3} \cdot \sqrt{3}}$

$$\begin{aligned} &= \frac{7\sqrt{3}}{2 \cdot 3} \\ &= \frac{7\sqrt{3}}{6} \end{aligned}$$

Sometimes, you can further simplify after rationalizing the denominator.

Example 3 – Simplify: a) $\frac{4 \cdot \sqrt{2}}{\sqrt{2} \cdot \sqrt{2}}$

$$\begin{aligned} &= \frac{4\sqrt{2}}{2} \\ &= 2\sqrt{2} \end{aligned}$$

b) $\frac{6\sqrt{3} \cdot \sqrt{18}}{\sqrt{18} \cdot \sqrt{18}}$

$$\begin{aligned} &= \frac{6\sqrt{54}}{18} \\ &= \frac{6\sqrt{9 \cdot 6}}{18} \\ &= \frac{18\sqrt{6}}{18} \\ &= \sqrt{6} \end{aligned}$$

c) $\frac{7\sqrt{6} \cdot \sqrt{10}}{2\sqrt{10} \cdot \sqrt{10}}$

$$\begin{aligned} &= \frac{7\sqrt{60}}{2 \cdot 10} \\ &= \frac{7\sqrt{4 \cdot 15}}{20} \\ &= \frac{14\sqrt{15}}{20} \\ &= \frac{7\sqrt{15}}{10} \end{aligned}$$

Sometimes, you can first simplify by dividing, and then you may or may not have to rationalize the denominator after. When dividing radicals, divide the coefficients, and divide the radicands.

Example 4 – Simplify a) $\frac{4\sqrt{14}}{2\sqrt{2}}$
 $= 2\sqrt{7}$

b) $\frac{-9\sqrt{48}}{6\sqrt{3}}$
 $= \frac{-3\sqrt{16}}{2}$
 $= \frac{-3(4)}{2}$
 $= \frac{-12}{2}$
 $= -6$

c) $\frac{6\sqrt{3}}{\sqrt{18}}$
 $\frac{6\sqrt{1}}{\sqrt{6}}$
 $\frac{6 \cdot 1}{\sqrt{6}}$
 $\frac{6}{\sqrt{6}}$
 $\frac{6 \cdot \sqrt{6}}{\sqrt{6} \cdot \sqrt{6}}$
 $= \frac{6\sqrt{6}}{6}$
 $= \sqrt{6}$

When solving quadratic equations using the quadratic formula (section 3.3), you will often have exact answers that may need simplifying. Here are some examples:

Example 5 – Simplify a) $\frac{-3 \pm \sqrt{27}}{6}$

b) $\frac{2 \pm \sqrt{48}}{2}$

c) $\frac{5 \pm \sqrt{8}}{10}$

a) $\frac{-3 \pm \sqrt{9 \cdot 3}}{6}$

(b) $\frac{2 \pm \sqrt{16 \cdot 3}}{2}$

(c) $\frac{5 \pm \sqrt{4 \cdot 2}}{10}$

$\frac{-3 \pm 3\sqrt{3}}{6}$

$\frac{2 \pm 4\sqrt{3}}{2}$

$\frac{5 \pm 2\sqrt{2}}{10}$

$= \frac{-1 \pm \sqrt{3}}{2}$

$= 1 \pm 2\sqrt{3}$

3.2A – Solving Quadratic Equations by Completing the Square

square
root
principle

When there is no bx term in a quadratic equation, first look to see if it is a *difference of squares* (is each term a perfect square, and is there a subtraction sign in between?). If it is not a difference of squares, it can be solved by the *square root principle*.

The general form of a quadratic equation that you could solve using the square root principle is $ax^2 \pm c = 0$.

Example 1 – Solve $9y^2 - 21 = 0$

- 1) Get everything that isn't squared to one side.
- 2) Square root both sides.
- 3) Consider both the positive and negative square roots.
- 4) Simplify any radical solutions as much as possible in exact form.

$$\textcircled{1} \quad 9y^2 - 21 = 0 \quad \textcircled{3} \quad y = \pm \frac{\sqrt{21}}{\sqrt{9}}$$

$$9y^2 = 21$$

$$y^2 = \frac{21}{9}$$

$$y = \pm \frac{\sqrt{21}}{3}$$

$$\textcircled{2} \quad \sqrt{y^2} = \pm \sqrt{\frac{21}{9}}$$

Example 2 - Solve: a) $2x^2 - 11 = 87$

b) $50y^2 = 72$

c) $(x+3)^2 = 16$

$$\textcircled{a} \quad \frac{2x^2}{2} = \frac{98}{2}$$

$$x^2 = 49$$

$$\sqrt{x^2} = \pm \sqrt{49}$$

$$x = \pm 7$$

$$\textcircled{b} \quad \frac{50y^2}{50} = \frac{72}{50}$$

$$y^2 = \frac{36}{25}$$

$$\sqrt{y^2} = \pm \sqrt{\frac{36}{25}}$$

$$y = \pm \frac{\sqrt{36}}{\sqrt{25}}$$

$$y = \pm \frac{6}{5}$$

$$\textcircled{c} \quad \sqrt{(x+3)^2} = \pm \sqrt{16}$$

$$x+3 = \pm 4$$

$$x+3 = 4 \quad x+3 = -4$$

$$x = 1 \quad x = -7$$

d) $3x^2 - 8 = 0$

e) $(x-1)^2 = 12$

$$\frac{3x^2}{3} = \frac{8}{3}$$

$$x^2 = \frac{8}{3}$$

$$\sqrt{x^2} = \pm \sqrt{\frac{8}{3}}$$

$$x = \pm \frac{\sqrt{8}}{\sqrt{3}}$$

$$x = \pm \frac{2\sqrt{2} \cdot \sqrt{3}}{\sqrt{3} \cdot \sqrt{3}}$$

$$x = \pm \frac{2\sqrt{6}}{3}$$

$$\textcircled{e} \quad \sqrt{(x-1)^2} = \pm \sqrt{12}$$

$$x-1 = \pm 2\sqrt{3}$$

$$x = \pm 2\sqrt{3} + 1$$

or

$$x = 1 \pm 2\sqrt{3}$$

completing
the square
when $a = 1$

Sometimes factoring quadratic equations (3.1) is not possible, as you cannot find the two numbers that multiply to c (or ac) and add to b . When this is the case, you can still solve the quadratic equation by a method called *completing the square*.

Example 3 – Solve $x^2 - 24 = -10x$ by completing the square. This example can easily be solved by factoring, but we will use it to introduce how to complete the square.

- 1) Get c to one side of the equation.
- 2) If the a value is 1, find the b value, halve it, and square it.
- 3) Add the number from step 2 to BOTH sides of the equation.
- 4) Factor the trinomial on the left (we created a perfect square trinomial, so it will lead to 2 brackets that are exactly the same).
- 5) Solve using the square root property.

$$x^2 - 24 = -10x$$

$$x+5 = \pm 7$$

$$\textcircled{1} x^2 + 10x = 24$$

$$x+5 = 7 \quad x+5 = -7$$

$$\textcircled{2} b = 10 ; 5 ; 25$$

$$x = 2 \quad x = -12$$

$$\textcircled{3} x^2 + 10x + 25 = 24 + 25$$

$$\textcircled{4} (x+5)^2 = 49$$

$$\textcircled{5} \sqrt{(x+5)^2} = \pm \sqrt{49}$$

Example 4 – Solve by completing the square. Express the solutions in exact form.

a) $w^2 - 4w - 11 = 0$

b) $x^2 + 5x + 7 = 0$

c) $m^2 - 5m + 3 = 0$

$$w^2 - 4w = 11$$

$$x^2 + 5x = -7$$

$$m^2 - 5m = -3$$

$$b = -4 ; -2 ; 4$$

$$b = 5, \frac{5}{2}, \frac{25}{4}$$

$$b = -5, -\frac{5}{2}, \frac{25}{4}$$

$$w^2 - 4w + 4 = 11 + 4$$

$$x^2 + 5x + \frac{25}{4} = \frac{-7 \times 4}{1 \times 4} + \frac{25}{4}$$

$$m^2 - 5m + \frac{25}{4} = \frac{-3 \times 4}{1 \times 4} + \frac{25}{4}$$

$$(w-2)^2 = 15$$

$$x^2 + 5x + \frac{25}{4} = \frac{-3}{4}$$

$$\left(m - \frac{5}{2}\right)^2 = \frac{13}{4}$$

$$w-2 = \pm \sqrt{15}$$

$$\left(x + \frac{5}{2}\right)^2 = \frac{-3}{4}$$

$$m - \frac{5}{2} = \pm \sqrt{\frac{13}{4}}$$

$$w = 2 \pm \sqrt{15}$$

cannot sq. root
right side, therefore
no solutions
(the parabola has
no x -ints)

$$m - \frac{5}{2} = \pm \frac{\sqrt{13}}{\sqrt{4}}$$

$$m = \pm \frac{\sqrt{13}}{2} + \frac{5}{2}$$

$$m = \frac{5 \pm \sqrt{13}}{2}$$

3.2B – Solving Quadratic Equations by Completing the Square Part 2

completing the square when $a \neq 1$

If $a \neq 1$, there are a few more considerations when *completing the square*.

Example 1 – Solve $2x^2 - 5x - 1 = 0$ by completing the square

- 1) Get c to one side of the equation.
- 2) Factor the a value out of the left side.
- 3) Divide both sides by the a value to leave $x^2 + bx$ on the left side of the equation. THEN find the b value, halve it, and square it.
- 4) Add the number from step 3 to BOTH sides of the equation.
- 5) Factor the resulting trinomial on the left.
- 6) Solve using the square root principle and answer in exact form.

$$2x^2 - 5x - 1 = 0 \quad \textcircled{4} \quad x^2 - \frac{5}{2}x + \frac{25}{16} = \frac{1}{2} + \frac{25}{16}$$

$$\textcircled{1} \quad 2x^2 - 5x = 1 \quad \textcircled{5} \quad \left(x - \frac{5}{4}\right)^2 = \frac{33}{16}$$

$$\textcircled{2} \quad 2\left(x^2 - \frac{5}{2}x\right) = 1 \quad \textcircled{6} \quad x - \frac{5}{4} = \pm \frac{\sqrt{33}}{\sqrt{16}}$$

$$\textcircled{3} \quad x^2 - \frac{5}{2}x = \frac{1}{2} \quad x = \frac{5 \pm \sqrt{33}}{4}$$

$$b = \frac{-5}{2}; \frac{-5}{4}; \frac{25}{16} \quad x = \frac{\pm \sqrt{33}}{4} + \frac{5}{4}$$

Example 2 – Solve by completing the square. Answer b to the nearest hundredth.

a) $3x^2 - 2 = -4x$

$$3x^2 + 4x = 2$$

$$3\left(x^2 + \frac{4}{3}x\right) = 2$$

$$x^2 + \frac{4}{3}x = \frac{2}{3}$$

$$b = \frac{4}{3}; \frac{4}{6} = \frac{2}{3}; \frac{4}{9}$$

$$x^2 + \frac{4}{3}x + \frac{4}{9} = \frac{2}{3} + \frac{4}{9}$$

$$\left(x + \frac{2}{3}\right)^2 = \frac{10}{9}$$

$$x + \frac{2}{3} = \pm \frac{\sqrt{10}}{\sqrt{9}}$$

$$x = \pm \frac{\sqrt{10}}{3} - \frac{2}{3}$$

$$x = \frac{-2 \pm \sqrt{10}}{3}$$

b) $-2x^2 - 3x + 7 = 0$

$$7 = 2x^2 + 3x$$

$$2x^2 + 3x = 7$$

$$2\left(x^2 + \frac{3}{2}x\right) = 7$$

$$x^2 + \frac{3}{2}x = \frac{7}{2}$$

$$b = \frac{3}{2}; \frac{3}{4}; \frac{9}{16}$$

$$x^2 + \frac{3}{2}x + \frac{9}{16} = \frac{7}{2} + \frac{9}{16}$$

$$\left(x + \frac{3}{4}\right)^2 = \frac{65}{16}$$

$$x + \frac{3}{4} = \pm \frac{\sqrt{65}}{\sqrt{16}}$$

$$x = \frac{-3 \pm \sqrt{65}}{4}$$

$$x = \frac{\pm \sqrt{65} - 3}{4} \quad x = 1.27, -2.77$$

An easy way to halve a fraction is to double the denominator.

Example 3 - Solve by completing the square: $3x^2 + 6x - 1 = 0$

$$3x^2 + 6x = 1$$

$$\frac{3(x^2 + 2x)}{3} = \frac{1}{3}$$

$$x^2 + 2x = \frac{1}{3}$$

$$b=2; 1; 1$$

$$x^2 + 2x + 1 = \frac{1}{3} + \frac{1^2}{1 \times 3}$$

$$(x+1)^2 = \frac{4}{3}$$

$$x+1 = \pm \sqrt{\frac{4}{3}}$$

$$x+1 = \pm \frac{2 \cdot \sqrt{3}}{\sqrt{3} \cdot \sqrt{3}}$$

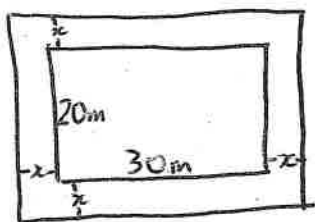
$$x+1 = \pm \frac{2\sqrt{3}}{3} - 1$$

$$x = \frac{\pm 2\sqrt{3}}{3} - \frac{3}{3}$$

$$x = \frac{-3 \pm 2\sqrt{3}}{3}$$

word
problems

Example 4 - Butchart Gardens wants to build a pathway around its rose garden. The rose garden is currently 30m x 20m. The pathway will be built by extending each side by an equal amount. If the area of the garden and path together is 1.173 times larger than the area of just the rose garden, how wide will the new path be (to the nearest tenth)?



$$A_{\text{rose garden}} = (20)(30) = 600 \text{ m}^2$$

$$A_{\text{garden + path}} = (600)(1.173) = 703.8 \text{ m}^2$$

$$\text{length of garden and path} = 30 + 2x$$

$$\text{width of garden and path} = 20 + 2x$$

$$(30 + 2x)(20 + 2x) = 703.8$$

$$600 + 60x + 40x + 4x^2 = 703.8$$

$$4x^2 + 100x + 600 = 703.8$$

$$4x^2 + 100x = 103.8$$

$$\frac{4(x^2 + 25x)}{4} = \frac{103.8}{4}$$

$$x^2 + 25x = 25.95$$

$$b = 25; 12.5; 156.25$$

$$x^2 + 25x + 156.25 = 25.95 + 156.25$$

$$(x + 12.5)^2 = 182.2$$

$$x + 12.5 = \pm 13.5$$

$$x = 1.0 \quad x = -26 \text{ reject}$$

The new path will
be 1.0m wide.

3.3 – The Quadratic Formula

Quadratic equations can be solved by graphing, factoring (3.1), the square root property (3.2) and/or completing the square (3.2). Each of these methods have advantages and limitations.

Any quadratic equation can be solved using something called the *quadratic formula*. If the quadratic equation is in standard form ($ax^2 \pm bx \pm c = 0$), the quadratic formula is:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example 1 - Solve $3x^2 + 5x - 2 = 0$ using the quadratic formula

$$a=3 \quad b=5 \quad c=-2$$

$$x = \frac{-5 \pm \sqrt{5^2 - 4(3)(-2)}}{2(3)} \quad x = \frac{-5 \pm \sqrt{49}}{6} \quad x = \frac{2}{6} = \frac{1}{3}$$

$$x = \frac{-5 \pm \sqrt{25+24}}{6} \quad x = \frac{-5 \pm 7}{6} \quad x = \frac{-12}{6} = -2$$

$$x = -2, \frac{1}{3}$$

discriminant

There could be 0, 1, or 2 resulting roots, depending on the **discriminant**, the expression under the square root ($b^2 - 4ac$). Solutions can be written in simplest radical form, or decimal form.

How would the discriminant determine the *nature of the roots* (the number of roots)?

if $b^2 - 4ac > 0$, there are 2 x-intercepts.

if $b^2 - 4ac = 0$, there is 1 x-intercept

if $b^2 - 4ac < 0$, there are no x-intercepts (cannot square root a negative number)

quadratic formula

Example 2 – Determine the nature of the roots, and then solve $3x^2 + 2x - 4 = 0$ using the quadratic formula

discriminant: $b^2 - 4ac$

$$= 2^2 - 4(3)(-4)$$

$$= 4 + 48$$

$$= 52$$

discrim > 0 , so 2 roots

$$x = \frac{-2 \pm \sqrt{2^2 - 4(3)(-4)}}{2(3)}$$

$$x = \frac{-2 \pm \sqrt{52}}{6}$$

$$a=3 \quad b=2 \quad c=-4$$

$$x = \frac{-2 \pm 2\sqrt{13}}{6}$$

$$x = \frac{-1 \pm \sqrt{13}}{3}$$

Example 3 - Determine the nature of the roots for $\frac{1}{4}x^2 - 3x + 9 = 0$

$$b^2 - 4ac$$

$$= (-3)^2 - 4(0.25)(9)$$

$$= 9 - 9$$

$$= 0 \quad \text{so } \underline{\underline{1 \text{ root}}}$$

$$a = 0.25 \quad b = -3 \quad c = 9$$

Example 4 - Solve using the quadratic formula. Leave answers in exact form.

a) $x^2 = 2x + 1$

$$x^2 - 2x - 1 = 0$$

$$a = 1, b = -2, c = -1$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(-1)}}{2(1)}$$

$$x = \frac{2 \pm \sqrt{4 + 4}}{2}$$

$$x = \frac{2 \pm \sqrt{8}}{2} \quad \left(x = 1 \pm \sqrt{2} \right)$$

$$x = \frac{2 \pm 2\sqrt{2}}{2}$$

b) $\frac{x^2}{2} - \frac{5x}{6} = \frac{-3}{2} \quad \frac{1}{2}x^2 - \frac{5}{6}x + \frac{3}{2} = 0$

$$a = \frac{1}{2} \quad b = -\frac{5}{6} \quad c = \frac{3}{2}$$

$$x = \frac{\frac{5}{6} \pm \sqrt{\left(-\frac{5}{6}\right)^2 - 4\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)}}{2\left(\frac{1}{2}\right)}$$

$$x = \frac{\frac{5}{6} \pm \sqrt{\frac{25}{36} - \frac{3 \times 36}{1 \times 36}}}{1}$$

$$x = \frac{5}{6} \pm \sqrt{-\frac{83}{36}}$$

No solutions

derivation

Derive the quadratic formula:

$$ax^2 + bx + c = 0$$

$$ax^2 + bx = -c$$

$$a\left(x^2 + \frac{b}{a}x\right) = -c$$

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

$$\frac{b}{a} ; \frac{b}{2a} ; \frac{b^2}{4a^2}$$

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{-c}{a} + \frac{b^2}{4a^2}$$

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{-4ac}{4a^2} + \frac{b^2}{4a^2}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm \sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \pm \frac{\sqrt{b^2 - 4ac}}{2a} - \frac{b}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

6.5 - Rational Equations

A rational equation is an equation containing at least one rational expression. Remember, when working with an equation, whatever you do to one side, you do to the other side.

Steps to solving rational equations:

- 1) Factor each denominator if possible.
- 2) Identify any undefined values (and do this throughout).
- 3) Multiply both sides of the equation by what would be the lowest common denominator in order to eliminate the fractions.
- 4) Solve the equation.
- 5) Check your solutions.

Example 1 - Solve a) $\frac{x}{4} + \frac{6}{x} = 3.5$ $x \neq 0$

$$\frac{x}{4} + \frac{6}{x} = 3.5$$

The common denominator would be $4x$, so multiply every term by $4x$

$$\frac{(4x)x}{4} + \frac{(4x)6}{x} = 3.5(4x)$$

$$x^2 + 24 = 14x$$

$$x^2 - 14x + 24 = 0$$

$$(x-12)(x-2) = 0$$

$$x = 12, 2$$

Check each in original, and they both work.

Example 2 - Solve $\frac{x}{x-5} - \frac{3}{x+1} = \frac{30}{x^2-4x-5}$

$$\frac{x}{x-5} - \frac{3}{x+1} = \frac{30}{(x-5)(x+1)}$$

$x \neq 5, -1$

$$\frac{x(x+1)}{x-5} - \frac{3(x-5)(x+1)}{x+1} = \frac{30(x-5)(x+1)}{(x-5)(x+1)}$$

$$x(x+1) - 3(x-5) = 30$$

$$x^2 + x - 3x + 15 = 30$$

$$x^2 - 2x + 15 = 30$$

b) $3 - \frac{6}{x} = x^2 + 8x$ $x \neq 0$

Multiply every term by x

$$3x - 6 = x^2 + 8x$$

$$x^2 + 5x + 6 = 0$$

$$(x+3)(x+2) = 0$$

$$x = -3, -2$$

Check:

$$x = -3$$

$$3 - \frac{6}{-3} = -3 + 8$$

$$3 + 2 = 5$$

✓

$$x = -2$$

$$3 - \frac{6}{-2} = -2 + 8$$

$$3 + 3 = 6$$

✓

$$x^2 - 2x - 15 = 0$$

$$(x-5)(x+3) = 0$$

$$x = \cancel{5}, -3$$

↑
undefined value

$$\underline{\underline{x = -3}}$$

Example 3 - Solve $\frac{3x}{x+2} - \frac{5}{x-3} = \frac{-25}{x^2-x-6}$

$$\frac{3x}{x+2} - \frac{5}{x-3} = \frac{-25}{(x-3)(x+2)}$$

$$x \neq -2, 3$$

$$\frac{3x \cdot \cancel{(x+2)(x-3)}}{\cancel{x+2} \cdot \cancel{x-3}} - \frac{5 \cdot \cancel{(x+2)(x-3)}}{\cancel{x-3} \cdot \cancel{x+2}} = \frac{-25 \cdot \cancel{(x+2)(x-3)}}{\cancel{(x-3)(x+2)}}$$

$$3x(x-3) - 5(x+2) = -25$$

$$3x^2 - 9x - 5x - 10 = -25$$

$$3x^2 - 14x + 15 = 0$$

$$3x^2 - 9x - 5x + 15 = 0$$

$$3x(x-3) - 5(x-3) = 0$$

$$(x-3)(3x-5) = 0$$

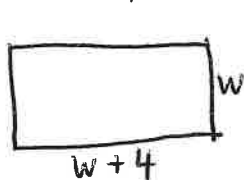
$$x = \cancel{x}, \frac{5}{3}$$

↑
undefined value

$$x = \frac{5}{3}$$

3.5 – Applications of Quadratic Equations

Example 1 – The area of a regulation Ping Pong table is 45ft^2 . The length is 4ft more than the width. What are the dimensions of the table?



$$45 = w^2 + 4w$$

$$w^2 + 4w - 45 = 0$$

$$(w+9)(w-5) = 0$$

$$w = -9, 5$$

reject

$$w = 5$$

$$w + 4$$

$$5 + 4$$

$$l = 9$$

The dimensions are
 $9\text{ft} \times 5\text{ft}$.

$$A = w(w+4)$$

$$45 = w(w+4)$$

Example 2 – The sum of a number and twice its reciprocal is $\frac{9}{2}$. Find the number.

Let $x =$ the number

$$x + 2\left(\frac{1}{x}\right) = \frac{9}{2}$$

$$x + \frac{2}{x} = \frac{9}{2} \quad \boxed{x \neq 0}$$

$$\overset{(2x)}{x} + \frac{2(2x)}{x} = \frac{9(2x)}{2}$$

$$2x^2 + 4 = 9x$$

$$2x^2 - 9x + 4 = 0$$

$$2x^2 - 8x - x + 4 = 0$$

$$2x(x-4) - 1(x-4) = 0$$

$$(x-4)(2x-1) = 0$$

$$x = 4, \frac{1}{2}$$

Check: $x = 4$

$$4 + 2\left(\frac{1}{4}\right) = \frac{9}{2}$$

$$4 + \frac{1}{2} = \frac{9}{2}$$

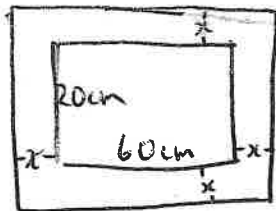
Check: $x = \frac{1}{2}$

$$\frac{1}{2} + 2\left(\frac{1}{\frac{1}{2}}\right) = \frac{9}{2}$$

$$\frac{1}{2} + 4 = \frac{9}{2}$$

The number can be 4 or $\frac{1}{2}$.

Example 3 – A 20cm by 60cm painting has a frame surrounding it. If the frame is the same width all around, and the total area of the frame is 516cm^2 , how wide is the frame?



$$A_{\text{painting}} = (20)(60) = 1200\text{cm}^2$$

$$A_{\text{frame}} = 516$$

$$A_{\text{total}} = 1200 + 516 = 1716\text{cm}^2$$

$$(60+2x)(20+2x) = 1716$$

$$1200 + 160x + 4x^2 = 1716$$

$$4x^2 + 160x - 516 = 0$$

$$4(x^2 + 40x - 129) = 0$$

$$43, -3$$

$$4(x+43)(x-3) = 0$$

$$x = -43, 3$$

$$x = 3\text{cm}$$

The frame is
 3cm wide.

Example 4 – Sally biked from Mt. Douglas to Stelly's, a distance of 25km, on two consecutive days. On Day 1, she rode 3km/h faster so her ride took 20 minutes less. Calculate her speed on both days and round to the nearest tenth.

Let x = speed on Day 2



	d (km)	s (km/hr)	time (hr)
Day 1	25	$x+3$	$\frac{25}{x+3}$
Day 2	25	x	$\frac{25}{x}$

20 mins = $\frac{1}{3}$ hr

$$\frac{25}{x+3} + \frac{1}{3} = \frac{25}{x} \quad \boxed{x \neq 0, -3}$$

$$75x + x(x+3) = 75(x+3)$$

$$75x + x^2 + 3x = 75x + 225$$

$$x^2 + 3x - 225 = 0$$

$a=1, b=3, c=-225$

$$x = \frac{-3 \pm \sqrt{3^2 - 4(1)(-225)}}{2}$$

$$x = \frac{-3 \pm \sqrt{909}}{2}$$

$$x = \frac{-3 \pm 30.15}{2}$$

$$x = -16.6, 13.6$$

Day 2 speed: 13.6 km/h
Day 1 speed: 16.6 km/h

Example 5 – The cold water tap can fill a container two hours faster than the hot water tap. The two taps together can fill the container in 80 minutes. How long does it take each tap to fill the container on its own?

Think back to the pattern of the equation for these types of questions:

$$\frac{\text{time it takes together}}{\text{time for solo}} + \frac{\text{time it takes together}}{\text{time for solo}} = 1$$

in minutes:

$$\frac{80}{x} + \frac{80}{x+120} = 1$$

↑ time for cold water solo ↑ time for hot water solo

$$80(x+120) + 80x = 1x(x+120)$$

$$80x + 9600 + 80x = x^2 + 120x$$

$$x^2 - 40x - 9600 = 0$$

-120, 80

$$(x-120)(x+80) = 0$$

$$x = 120, -80 \text{ reject}$$

$$x = 120 \text{ minutes}$$

$$x+120 = 120+120 = 240 \text{ mins}$$

Cold water solo takes 120 mins or 2 hrs
Hot water solo takes 240 mins or 4 hrs.

