

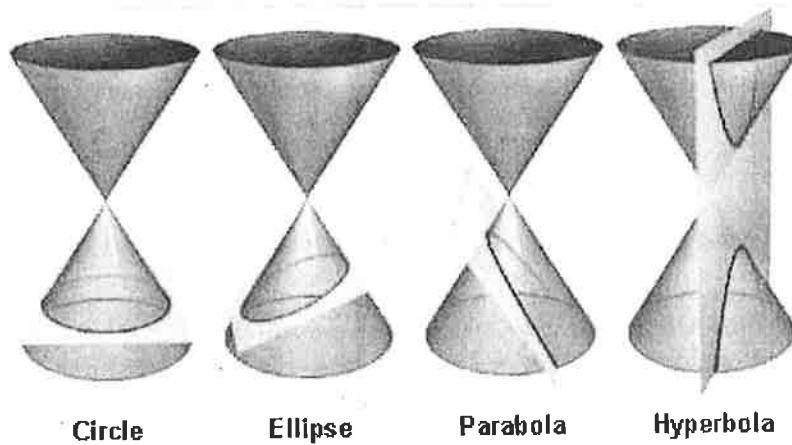
# Conic Sections (Conics)

## OCCURRENCE OF THE CONICS

Mathematicians have a habit of studying, just for the fun of it, things that seem utterly useless; then centuries later their studies turn out to have enormous scientific value.

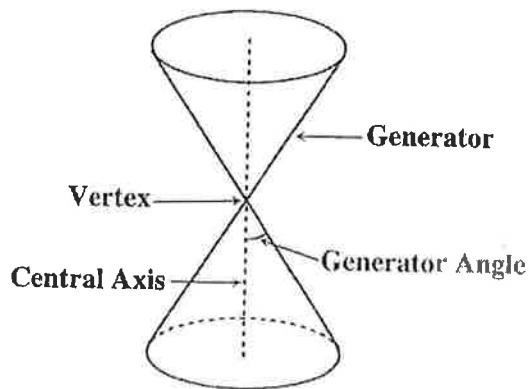
There is no better example of this than the work done by the ancient Greeks on the curves known as the conics: the ellipse, the parabola, and the hyperbola. They were first studied by one of Plato's pupils. No important scientific applications were found for them until the 17th century, when Kepler discovered that planets move in ellipses and Galileo proved that projectiles travel in parabolas.

Appollonius of Perga, a 3rd century B.C. Greek geometer, wrote the greatest treatise on the curves. His work "Conics" was the first to show how all three curves, along with the circle, could be obtained by slicing the same right circular cone at continuously varying angles.



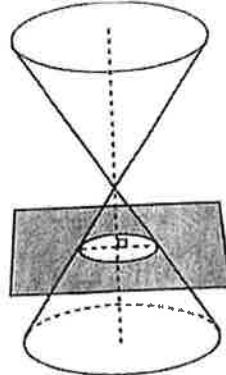
i.e. a curve developed from the intersection of a plane with a double right circular cone (double-napped cone) is referred to as a "conic section".

## Some terminology:

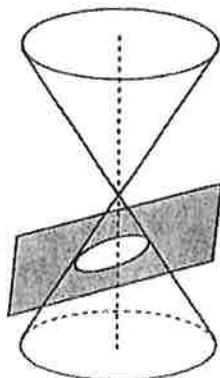


## Scenarios:

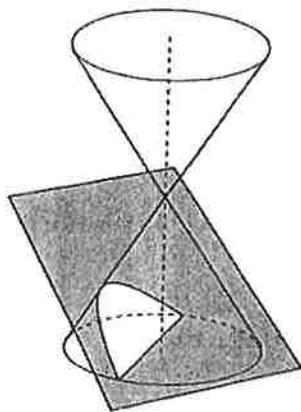
- i) If a plane cuts completely through one cone and is PERPENDICULAR to the central axis, then the curve is a CIRCLE.



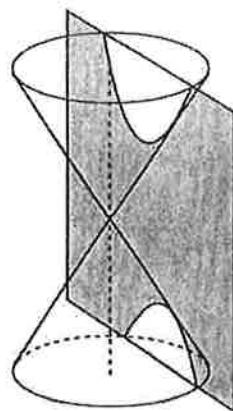
- ii) If a plane cuts completely through one cone and is NOT perpendicular to the central axis, then the curve is an ELLIPSE.



iii) If a plane does not cut completely through one cone, then the intersecting curve is a PARABOLA.

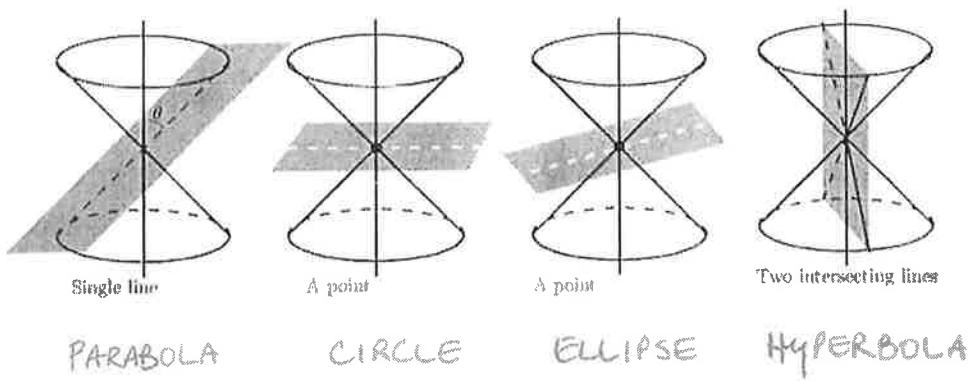


iv) If a plane cuts through BOTH cones, not at the vertex, then the intersection creates a HYPERBOLA.



\* note: any plane passing through the vertex produces a degenerate equation:

- Enrichment {
- i) circle  $\rightarrow$  point
  - ii) ellipse  $\rightarrow$  point
  - iii) parabola  $\rightarrow$  line
  - iv) hyperbola  $\rightarrow$  intersecting lines
- } see diagrams next page



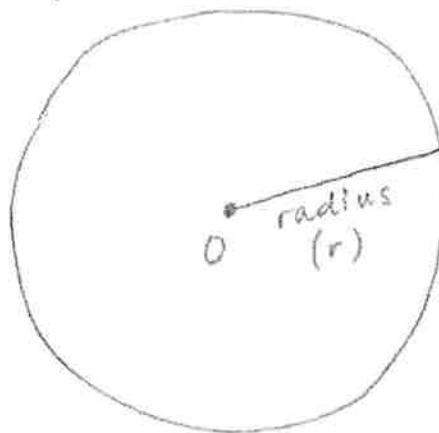
See <http://britton.disted.camosun.bc.ca/jbconics.htm>  
 for more about "real life" conics  
occurrences.

## 8.2 Circles

A circle is the set or locus ( ) of all points in a plane which are \_\_\_\_\_ from a fixed point.

- this "fixed point" is called the \_\_\_\_\_ (centre) of the circle.

The fixed distance from the centre to any point on the circle is called the \_\_\_\_\_.

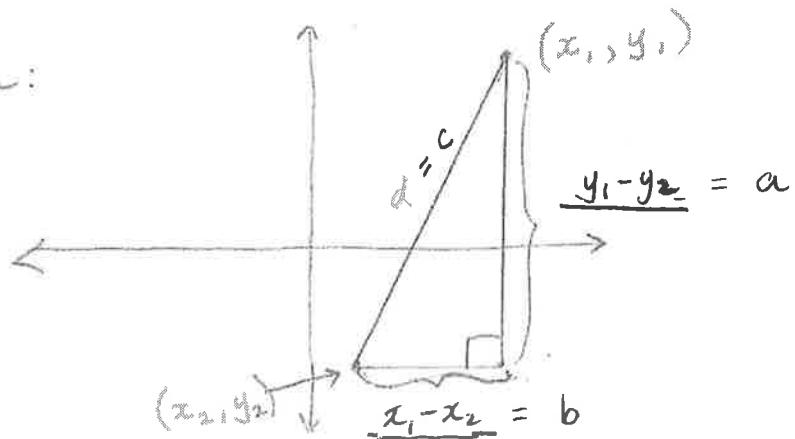


Knowing the centre's coordinates and any point on the circle's 'edge' can allow for the radius to be calculated using the distance formula:

$$d =$$

## DERIVATION of DISTANCE FORMULA :

Given:



$$a^2 + b^2 = c^2 \quad \text{PYTHAGOREAN THEOREM}$$

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 = d^2$$

$$d = \sqrt{\underbrace{(x_1 - x_2)^2}_{\text{}} + \underbrace{(y_1 - y_2)^2}_{\text{}}}$$

the order of the subtraction  
is irrelevant since the result  
is squared.

So, given a circle with centre  $(x_1, y_1)$   
and possessing point  $(x_2, y_2)$ :

$$r =$$

OR

$$r =$$

Ex1: Find the radius of a circle with centre  $(2, 3)$  and possessing point  $(5, 7)$ .

3) The STANDARD FORM of a circle with centre  $(h, k)$  and radius  $r$  is :

\* with respect to 'transformation form',  
$$(y = a + b(x - c)) + d)$$

$h$  represents \_\_, and  $k$  represents \_\_.

If a circle's centre is the graph's origin  $(0, 0)$ , then the equation is:

i.e. the 'BASIC' circle.

eg2: Write the standard form equation of the circle with centre  $(-3, 5)$  and  $r = 6$ .

The GENERAL FORM of a circle is obtained by expanding the standard form:

$$(x-h)^2 + (y-k)^2 = r^2$$

EXPAND:  $x^2 - 2hx + h^2 + y^2 - 2ky + k^2 = r^2$

RE-ARRANGE:  $x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - r^2 = 0$

General Form:  $Ax^2 + Cy^2 + Dx + Ey + F = 0$

where \_\_\_\_\_, and...

$$D = \underline{\hspace{2cm}} \quad E = \underline{\hspace{2cm}}$$

$$F = \underline{\hspace{2cm}}$$

eg3: Convert eg<sup>2</sup> to general form.

Check:

eg4: Write the equation of the circle  
(in both standard and general form)  
with centre  $(4, -1)$  and passing  
through  $(3, 7)$ .

In order to convert a given general form to standard form, you will need to COMPLETE THE SQUARE for both  $x$  and  $y$ :

$$x^2 + y^2 + Dx + Ey + F = 0$$

$$x^2 + Dx + y^2 + Ey = -F$$

$$(x^2 + Dx + \left(\frac{D}{2}\right)^2) + (y^2 + Ey + \left(\frac{E}{2}\right)^2) = -F + \left(\frac{D}{2}\right)^2 + \left(\frac{E}{2}\right)^2$$

$$\left(x + \frac{D}{2}\right)^2 + \left(y + \frac{E}{2}\right)^2 = -F + \left(\frac{D}{2}\right)^2 + \left(\frac{E}{2}\right)^2$$

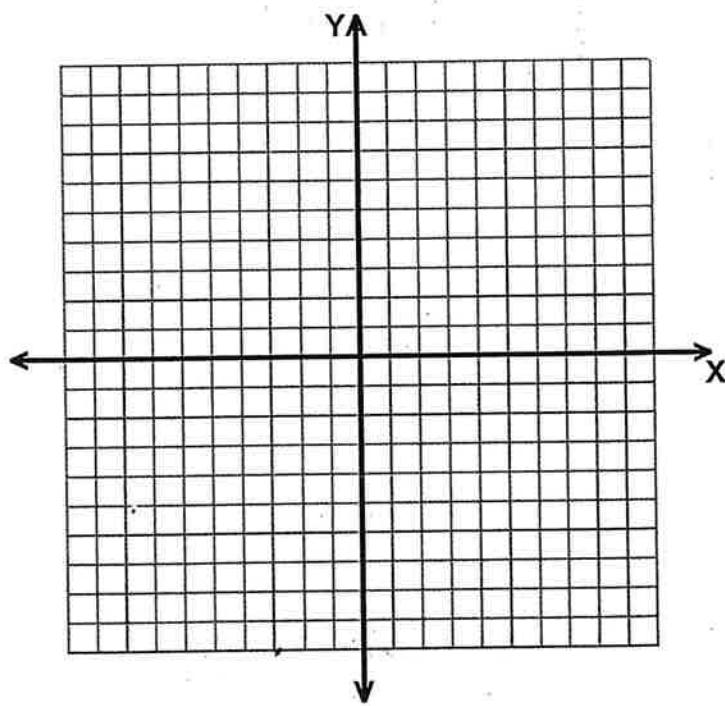
CENTRE:  $\left(-\frac{D}{2}, -\frac{E}{2}\right)$        $r = \sqrt{-F + \left(\frac{D}{2}\right)^2 + \left(\frac{E}{2}\right)^2}$

eg5: Given the general form of a circle's equation, find the centre and radius of the circle:

a)  $x^2 + y^2 + 14x - 72 = 0$

$$b) \quad x^2 + y^2 - x + 2y - \frac{59}{4} = 0$$

eg6: Graph (sketch)  $(x-5)^2 + (y+1)^2 = 9$



Do "The Circle" #1-45

and/or p. 360 #1-13

## 8.4 Ellipses

An ELLIPSE is the locus of all points in a plane such that the \_\_\_\_\_ of the distances from two given points in the plane, the \_\_\_\_\_, to any point on the ellipse is constant.

A 'horizontal' ellipse:

A 'vertical' ellipse:

\* foci exist upon \_\_\_\_\_ axis

### Variable Denotations

$a$  = half the length of the \_\_\_\_\_ axis

$b$  = half the length of the \_\_\_\_\_ axis

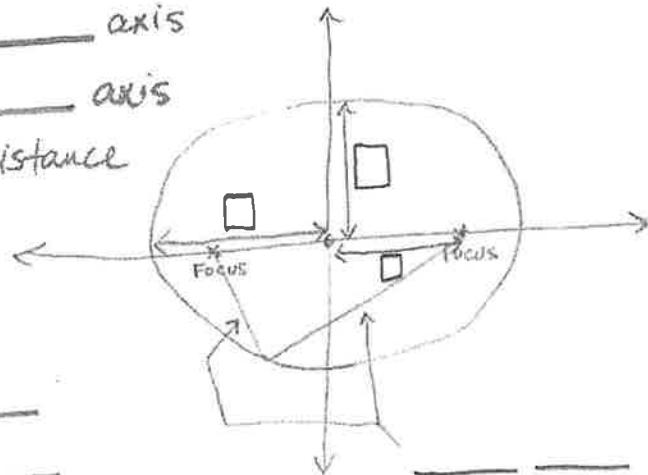
$c$  = half the length of the distance between the \_\_\_\_\_

(ie. distance from focus to centre)

So, major axis length = \_\_\_\_\_

minor axis length = \_\_\_\_\_

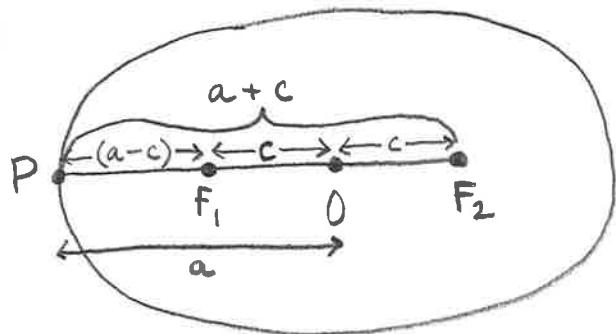
dist. btwn. foci = \_\_\_\_\_



FACT: The sum of the distances from the two foci to a point on the ellipse is equal to the length of the major axis ( $2a$ ).

i.e. The sum of the lengths of the FOCAL RADII equals the length  $2a$ .

PROOF:



$F_1, F_2$  : FOCI

O : CENTRE

P : POINT ON ELLIPSE

$$(a-c) + (a+c) = 2a \quad \text{major axis}$$

$\text{P to } F_1$        $\text{P to } F_2$

$$a + a = 2a$$

$2a = 2a$

Pythagorean Relationship between  $a$ ,  $b$ , and  $c$

Given an ellipse with major axis =  $2a$ , minor axis =  $2b$ , and distance between foci =  $2c$

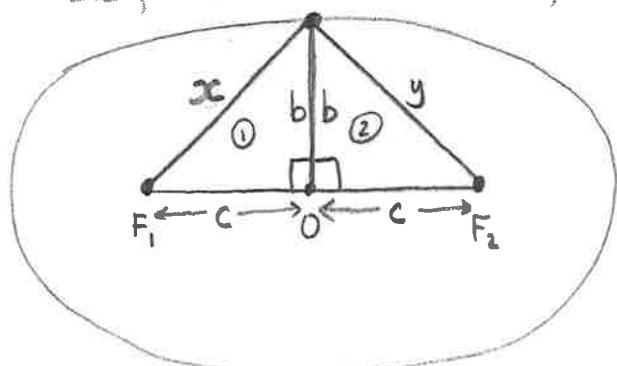
$$\Delta 1 \cong \Delta 2$$

Congruent  $\Delta$ s (SSS)

$$\text{so, } x = y$$

$$\text{From FACT above: } x + y = 2a$$

$$\text{so, } x = y = a$$



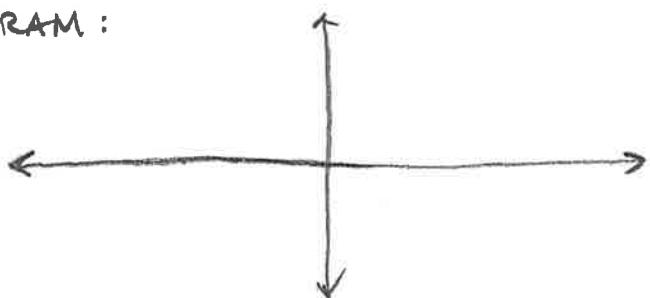
'Horizontal Ellipse'  $\rightarrow$  an ellipse with its major axis existing horizontally.

'Vertical Ellipse'  $\rightarrow$  an ellipse with its major axis existing vertically.

The STANDARD FORM of a horizontal ellipse, centred at  $(h, k)$ , with major axis of length  $2a$  and minor axis of length  $2b$  is:

- \* If the centre is the graph's origin,  $(0, 0)$ , then the equation simplifies to:

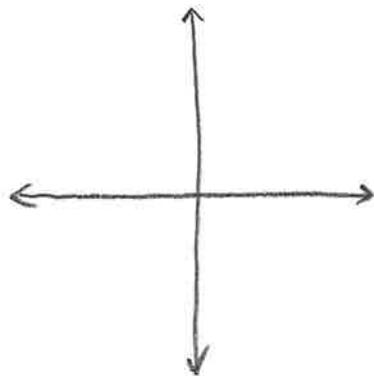
DIAGRAM:



The STANDARD FORM of a vertical ellipse, centred at  $(h, k)$ , with major axis of length  $2a$  and minor axis of length  $2b$  is:

- \* If the centre is the graph's origin,  $(0,0)$ , then the equation simplifies to:

DIAGRAM:



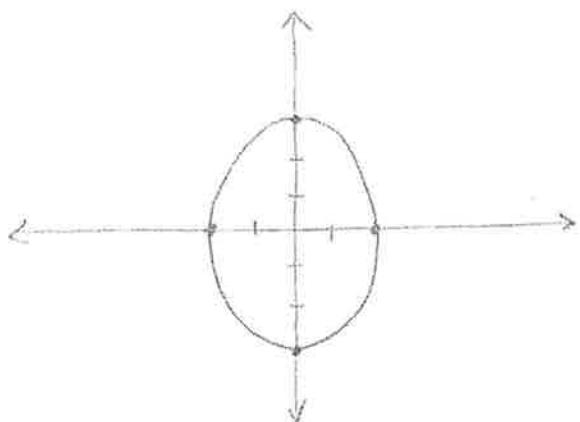
The GENERAL FORM of an ellipse with its axes parallel to the x- and y-axis is:

- \* just expand and simplify standard form.

eg 1: Write the equation for each ellipse  
in both standard and general form:

- a) CENTRE  $(0, 0)$  Foci @  $(3, 0)$  and  $(-3, 0)$   
 $d$  (major axis) = 10       $d$  (minor axis) = 8.

b)



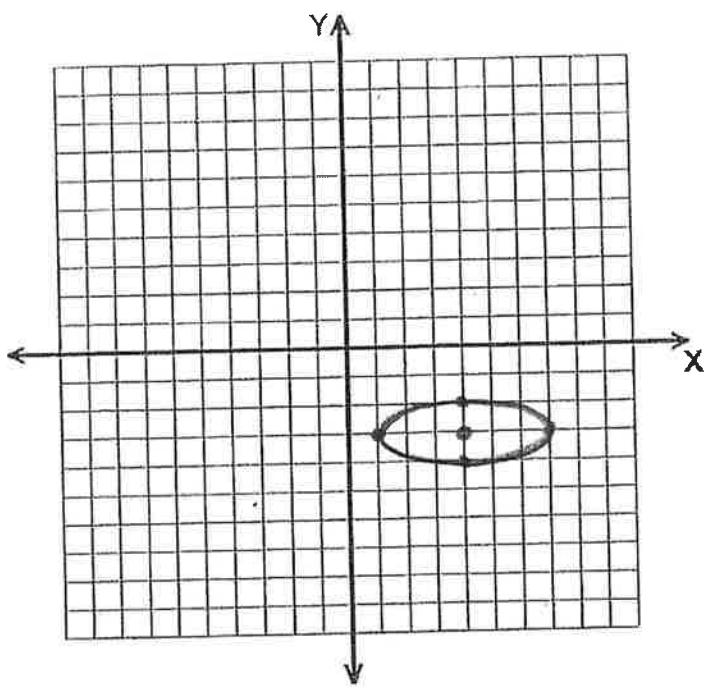
\* note: from ② and ③

Recall General form:  $Ax^2 + Cy^2 + Dx + Ey + F = 0$

When  $A < C \rightarrow$         ellipse

When  $A > C \rightarrow$         ellipse

c)



d) Centre  $(-2, 5)$  and passing through  $(-5, 5)$ ,  
 $(1, 5)$ ,  $(-2, -2)$ , and  $(-2, 12)$ .

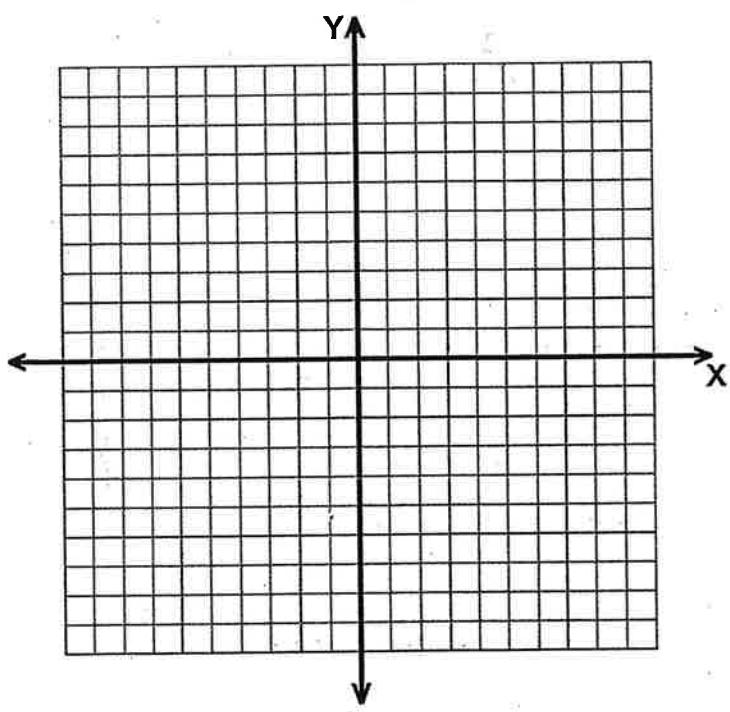
eg2: Given the following general form equations,  
find the coordinates of the centre and foci.  
Also, determine the length of the major axis  
and the minor axis.

a)  $x^2 + 2y^2 - 2x + 4y - 1 = 0$

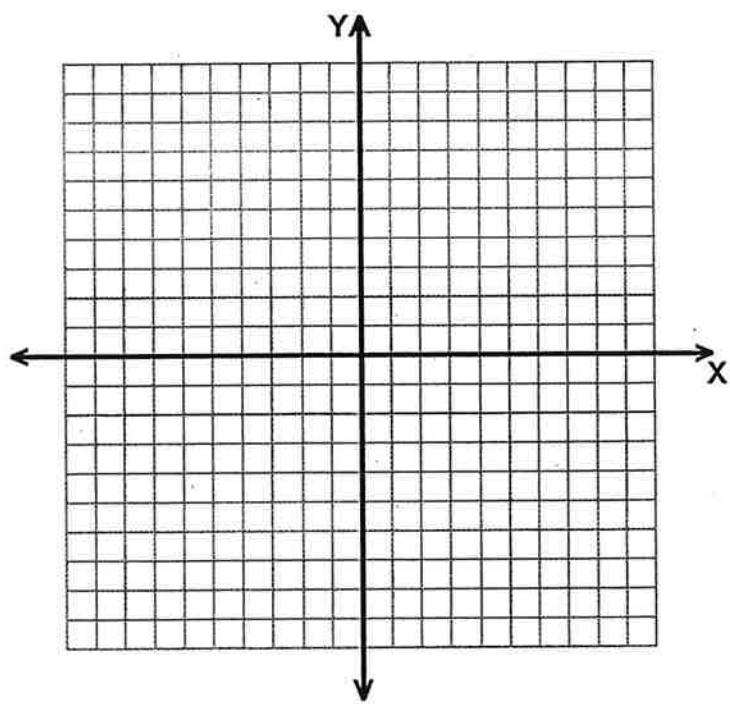
$$b) \quad 9x^2 + 4y^2 + 6x + 4y - 23 = 0$$

eg3. Sketch each ellipse:

a)  $4x^2 + (y-1)^2 = 64$



$$b) \frac{(x+5)^2}{4} + 4(y-1)^2 = 1$$

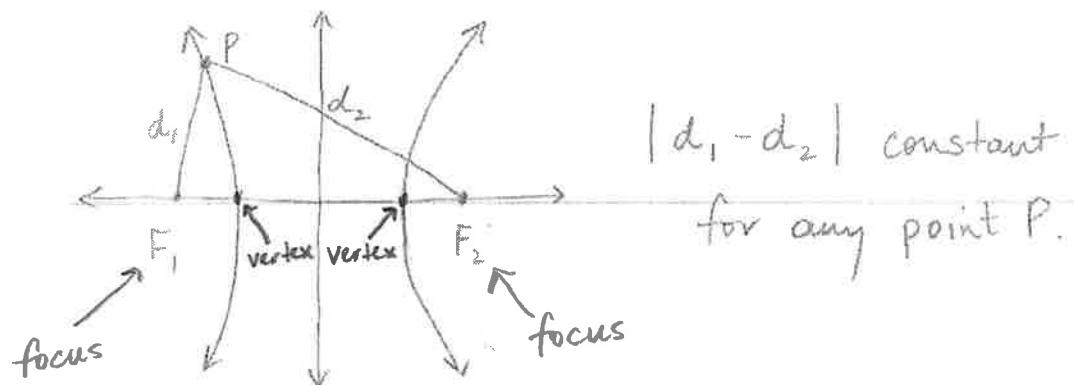


Do "The Ellipse" # 1-27, and/or p.373  
# 1-13

## 8.5 Hyperbolas

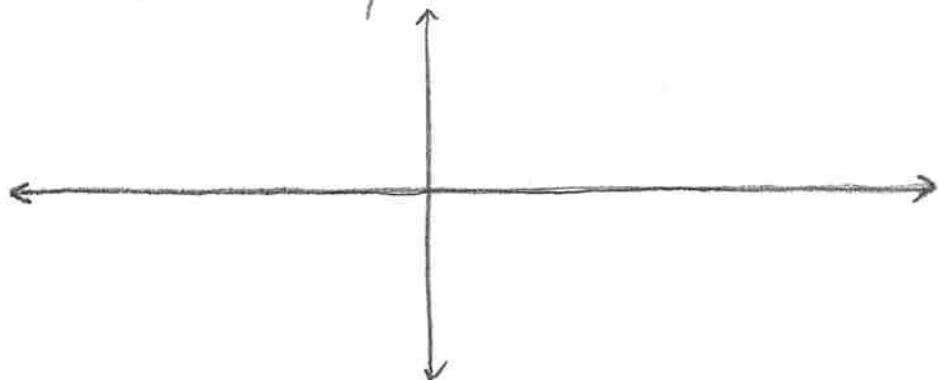
A hyperbola is the locus of all points such that the absolute value of the difference of the distances from any point on the hyperbola to two given points on the plane, the           , is constant.

i.e.

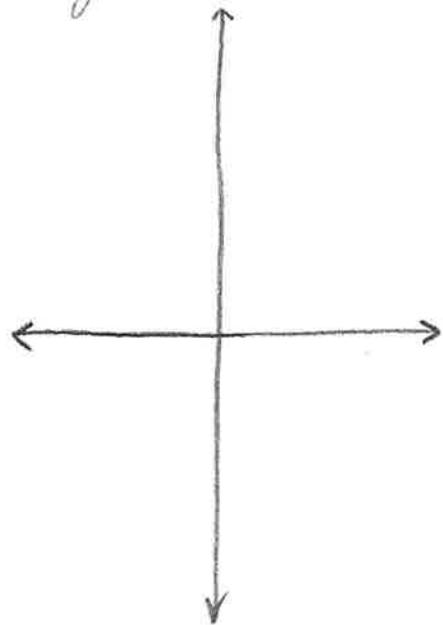


A hyperbola possesses two lines of symmetry. One intersects the hyperbola at its vertices (the            axis) and the other is perpendicular to the transverse axis (the            axis).

e.g. A 'horizontal' hyperbola:



e.g. A 'vertical' hyperbola:



The length of the transverse axis 'segment'  
is the distance between the two vertices.

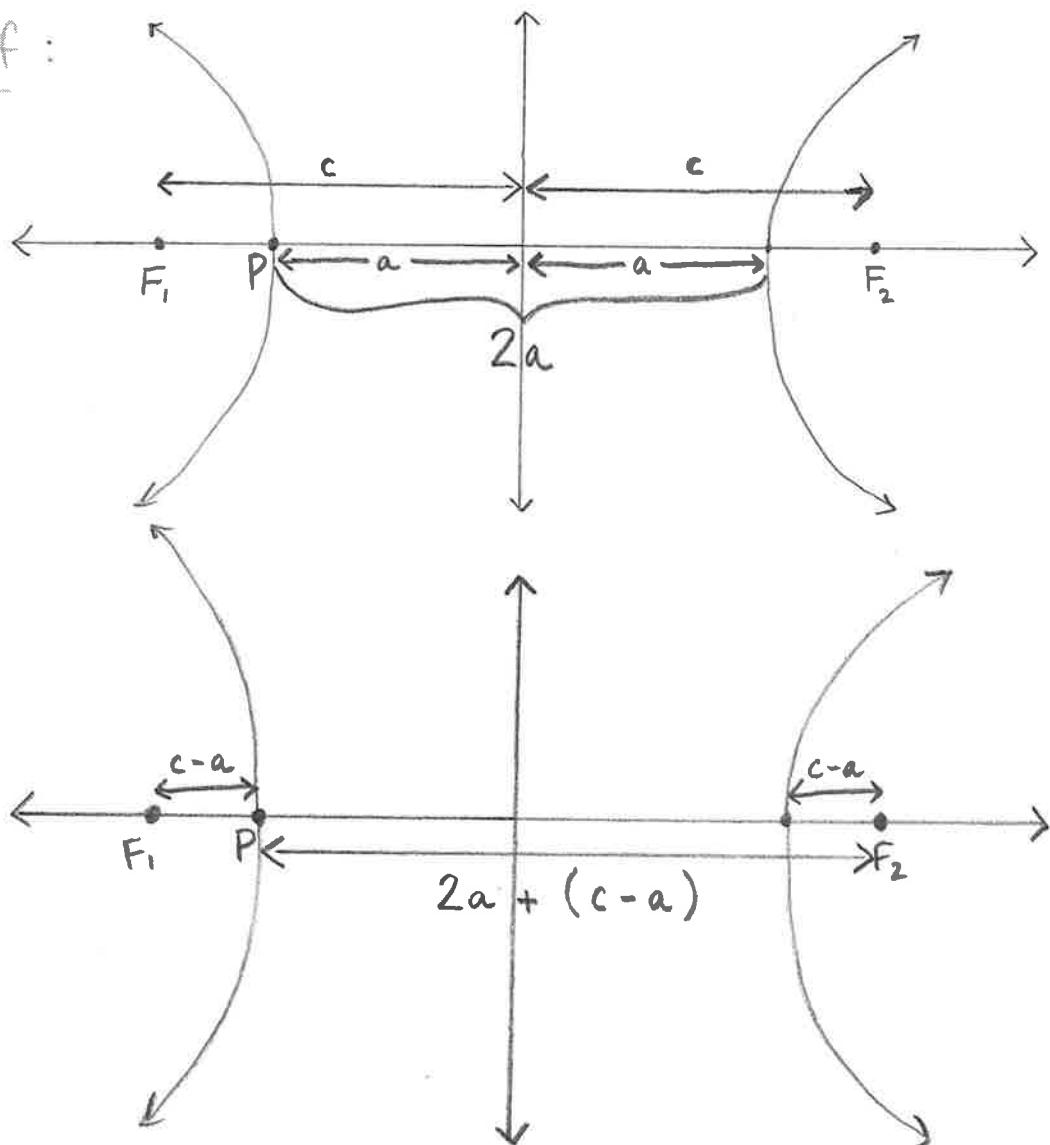
- length of transverse axis 'segment' = —
- half the length? —

The distance between the foci = —  
- half the distance? —

The length of the conjugate axis 'segment'  
(more on the derivation of this later) = —  
- half the distance? —

Fact: The absolute value of the difference of the focal radii (the distances from each focus to a point P on the hyperbola) is equal to the length of the transverse axis 'segment',  $2a$ .

Proof:



$$\begin{aligned}
 |F_1P - F_2P| &= |(c-a) - (2a + (c-a))| \\
 &= |(c-a) - (a+c)| \\
 &= |-2a| \\
 &= \boxed{2a} \quad \checkmark
 \end{aligned}$$

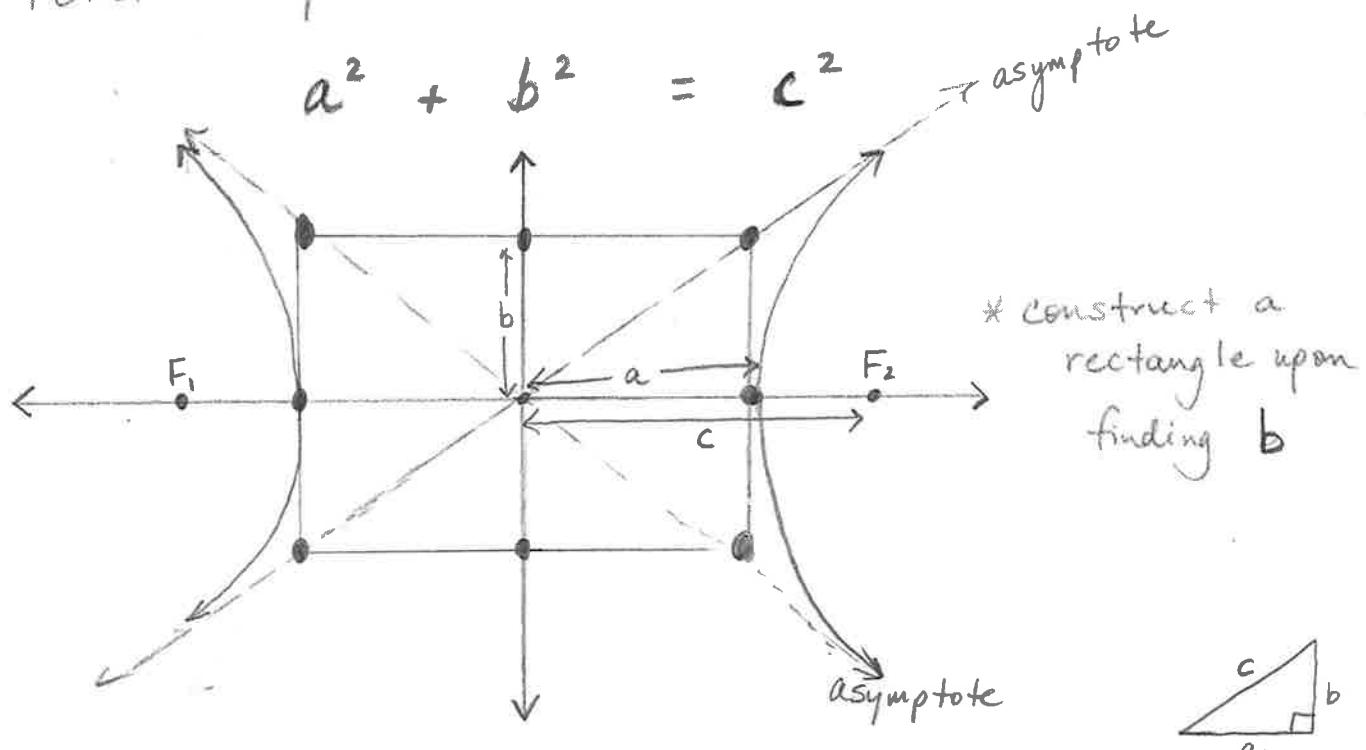
## Pythagorean Relationship

$a$  = half the length of the Transverse axis segment

$b$  = half the length of the Conjugate axis segment.

$c$  = half the distance between the foci.

For a hyperbola, the value of  $b$  is determined using the Pythagorean relationship:



Thus, the diagonal segment of the rectangle = \_\_\_\_\_

Also, the diagonals of the rectangle represent the hyperbola's \_\_\_\_\_, one with a slope of  $\boxed{\phantom{0}}$ , and one with a slope of  $\boxed{\phantom{0}}$ .

Two types of hyperbolas and their standard forms:

i) Transverse axis along the  $x$ -axis

\* 'horizontal' hyperbola

ii) Transverse axis along the  $y$ -axis

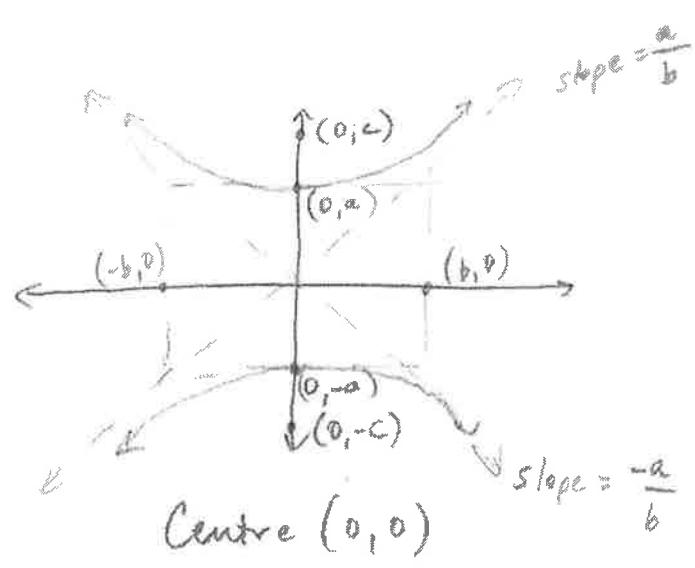
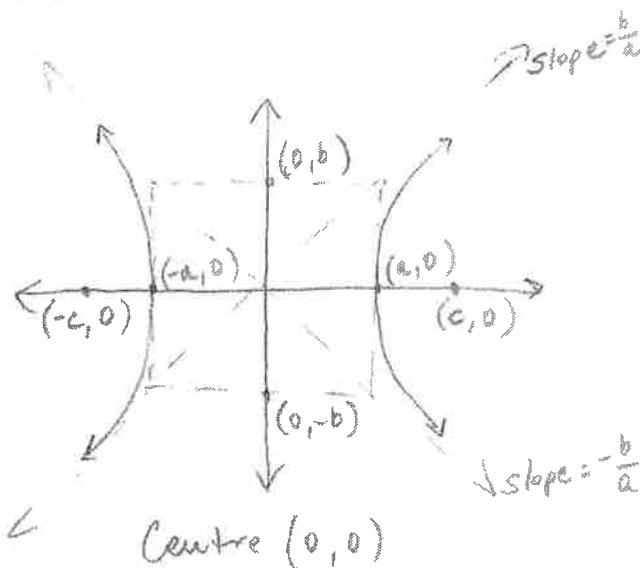
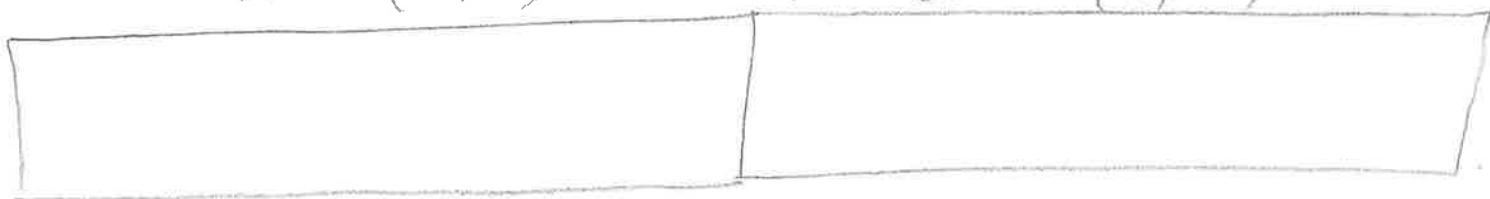
\* 'vertical' hyperbola

Standard form:

Centre  $(h, k)$

Standard form:

Centre  $(h, k)$



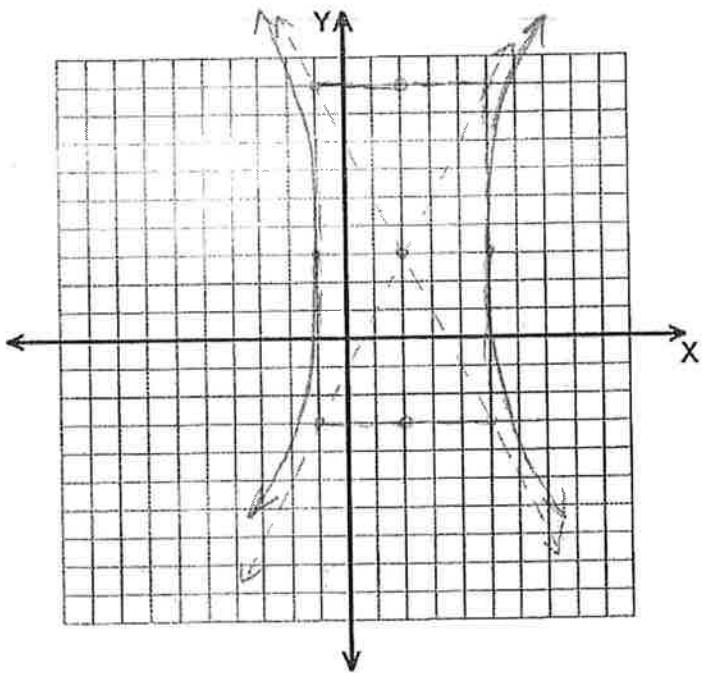
\*NOTE: a \_\_\_\_\_ hyperbola has perpendicular asymptotes;

General Form?

$\therefore$  \_\_\_\_\_ for rectangular hyperbolas.

eql: Given the following hyperbola, describe / provide :

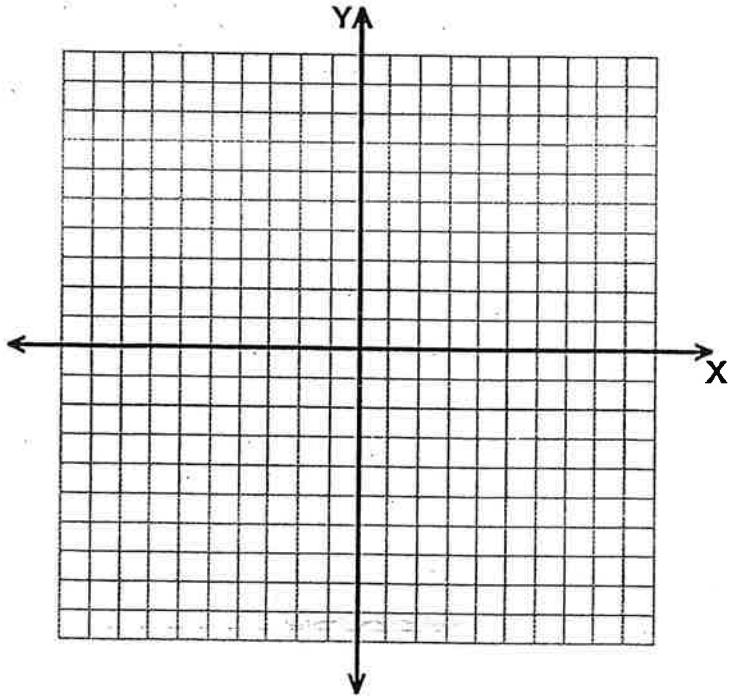
- a) its centre
- b) the asymptote equations
- c) the direction and length of the transverse axis
- d) the direction and length of the conjugate axis.
- e) the coordinates of the foci.
- f) the equation of the hyperbola in both standard and general form.



eg2: Given the hyperbola  $4x^2 - y^2 - 16x - 14y - 34 = 0$ ,  
find the centre, the coordinates of the vertices,  
the coordinates of the foci, and  
the equations of the asymptotes.

more space on  
next page →

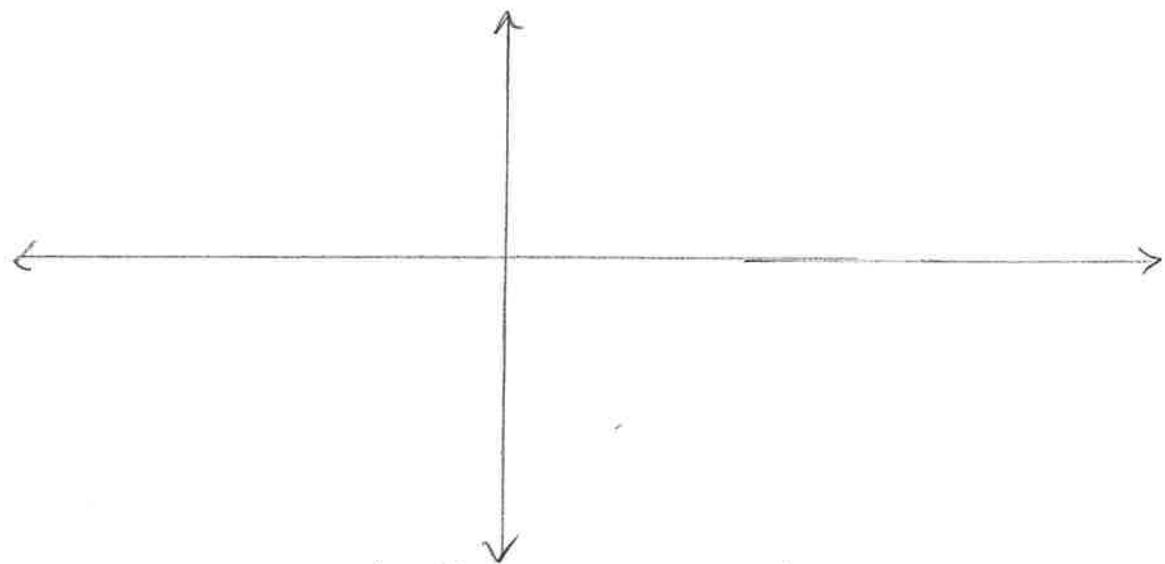
y3: Graph  $\frac{(y+2)^2}{4} - \frac{(x-1)^2}{25} = 1$



Do "The Hyperbola" # 1-36  
and/or p. 380 # 1-14

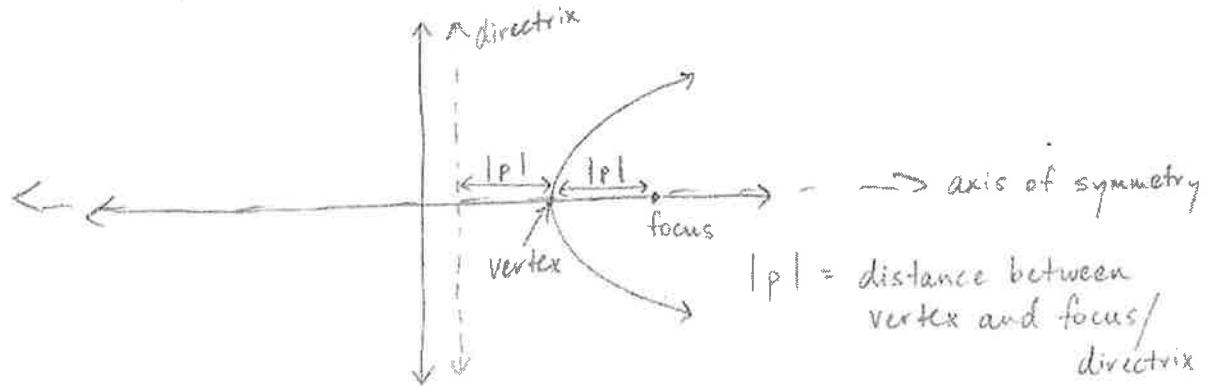
### 8.3 Parabolas

A parabola is the locus of all points in a plane that are the same distance from a line in the plane, the \_\_\_\_\_, as they are from a fixed point in the plane, not on the directrix, the \_\_\_\_\_.



- the VERTEX of the parabola lies on a perpendicular line segment that connects the FOCUS to the DIRECTRIX. The vertex lies on this segment.

- the extension of this line segment represents the parabola's \_\_\_\_\_.



For a parabola with an axis of symmetry that is parallel to the  $y$ -axis and vertex  $(h, k)$ :

- the equation of the axis of symmetry is \_\_\_\_\_

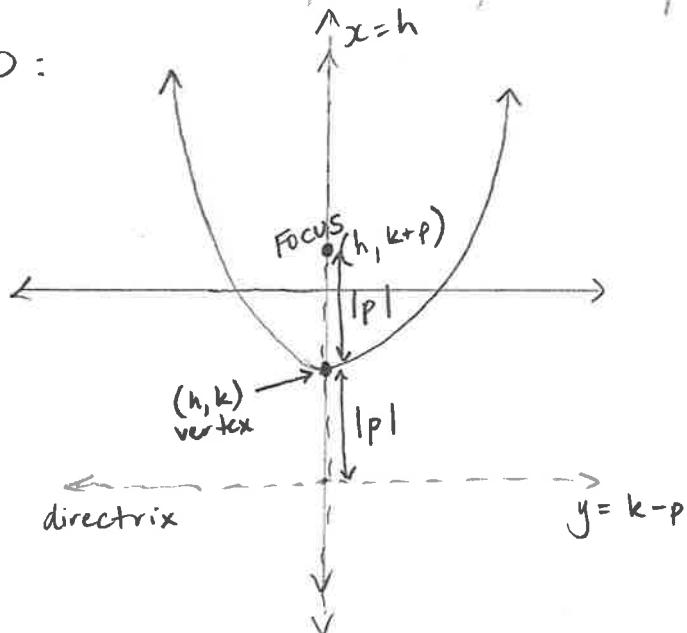
- the coordinates of the focus are \_\_\_\_\_

- the equation of the directrix is \_\_\_\_\_

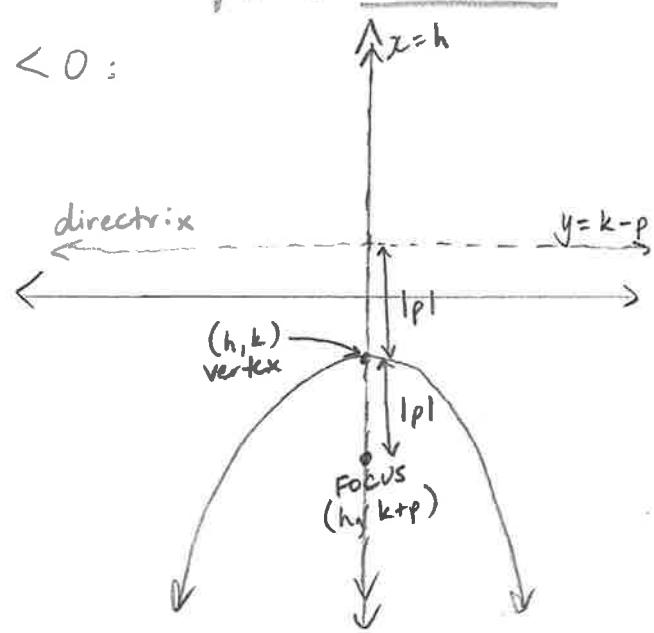
- when  $p > 0$ , the parabola opens \_\_\_\_\_

- when  $p < 0$ , the parabola opens \_\_\_\_\_

$p > 0$ :



$p < 0$ :



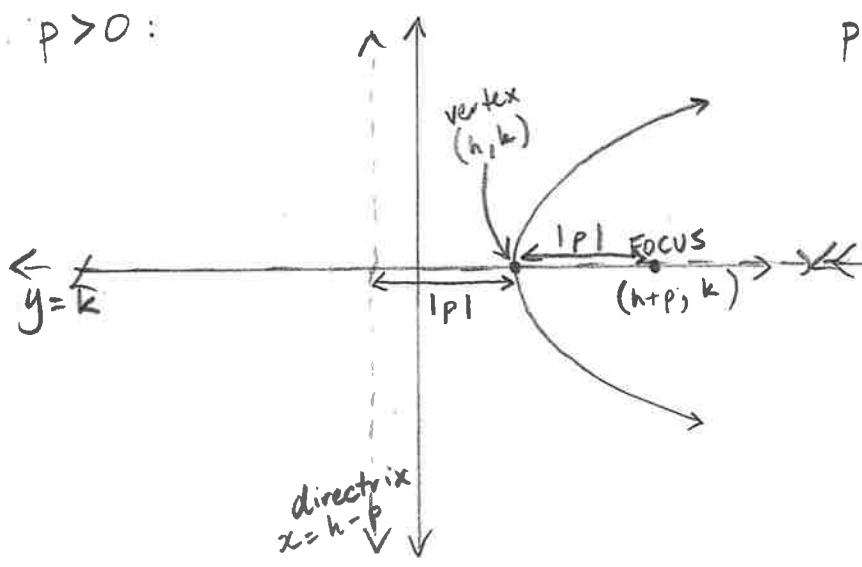
The STANDARD FORM of these parabolas is:

\* compare to  $y = a(x-h)^2 + k$  (Math 11)

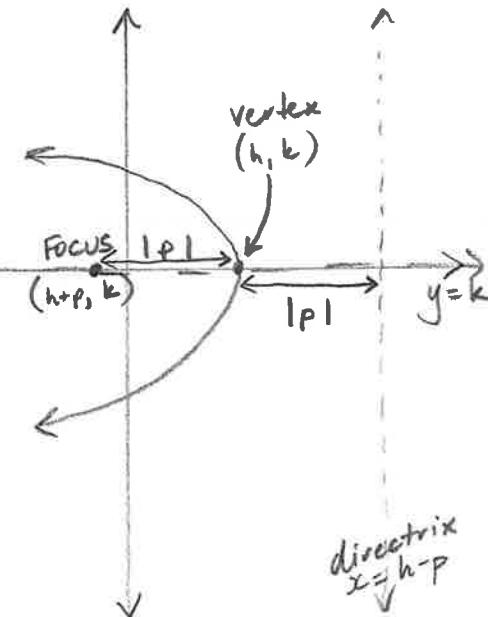
For a parabola with an axis of symmetry that is parallel to the  $x$ -axis and having vertex  $(h, k)$ :

- the equation of the axis of symmetry is \_\_\_\_\_
- the coordinates of the focus are \_\_\_\_\_
- the equation of the directrix is \_\_\_\_\_
- when  $p > 0$ , parabola opens \_\_\_\_\_
- when  $p < 0$ , parabola opens \_\_\_\_\_

$p > 0$ :

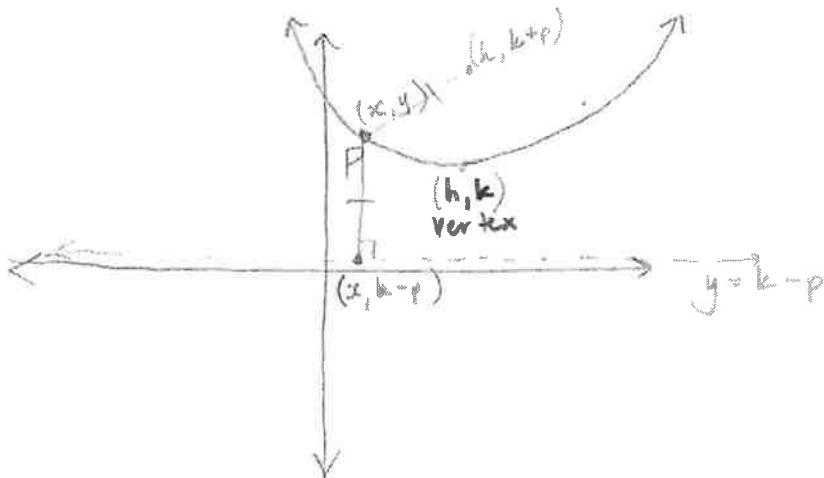


$p < 0$ :



The STANDARD FORM of these parabolas is:

## Derivation of $(x-h)^2 = 4p(y-k)$



\* The distance from point  $P(x, y)$  to the focus and directrix, respectively, must equal each other:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$\left. \begin{aligned} d \left( \overset{\text{POINT } P}{(x, y)} \text{ to focus} \right) &= \sqrt{(x-h)^2 + (y-(k+p))^2} \\ d \left( \overset{\text{POINT } P}{(x, y)} \text{ to directrix} \right) &= \sqrt{(x-x)^2 + (y-(k-p))^2} \end{aligned} \right\} \text{these are equal to each other.}$$

$$\sqrt{(x-h)^2 + (y-(k+p))^2} = \sqrt{(x-x)^2 + (y-(k-p))^2}$$

$$(x-h)^2 + (y-(k+p))^2 = (x-x)^2 + (y-(k-p))^2$$

$$(x-h)^2 + (y^2 - 2y(k+p) + (k+p)^2) = 0 + (y^2 - 2y(k-p) + (k-p)^2)$$

$$(x-h)^2 = (y^2 - 2y(k-p) + (k-p)^2) - (y^2 - 2y(k+p) + (k+p)^2)$$

$$(x-h)^2 = -2yk + 2yp + k^2 - 2kp + p^2 + 2yk + 2yp - k^2 - 2kp - p^2$$

$$(x-h)^2 = 4yp - 4kp$$

$$\boxed{(x-h)^2 = 4p(y-k)} \quad \checkmark$$

The GENERAL FORM of a parabola is :

where either

If  $C=0$ , parabola opens

If  $A=0$ , parabola opens

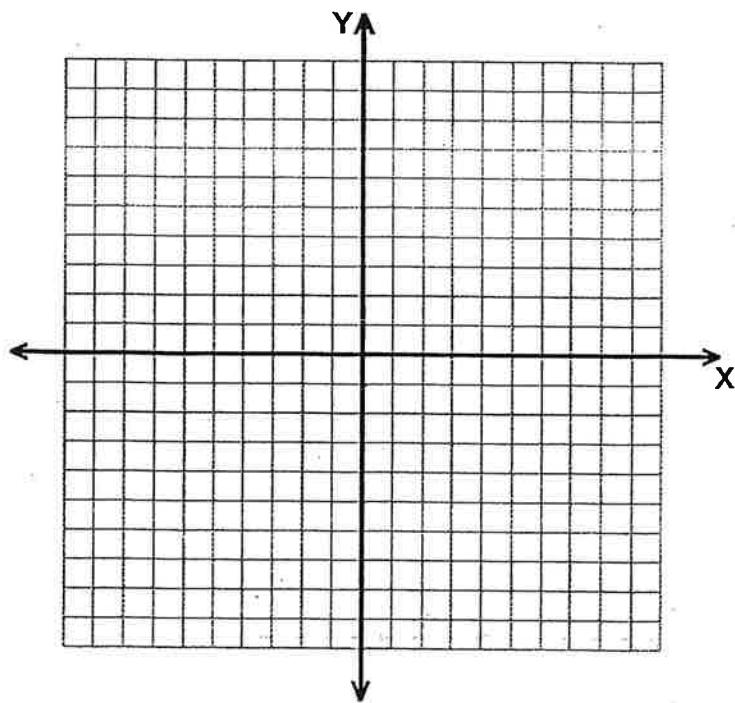
eg1: Write the equation of the parabola for each of the following in both STANDARD and GENERAL form:

a) focus @  $(2, 5)$ ; directrix @  $x = 6$

b) passes through  $(5, 2)$ , vertex @  $(4, 3)$ , opens down.

eg2: Find the coordinates of the focus and the vertex, the equations of the directrix and the axis of symmetry, and the direction of the opening of  $y^2 - 4x + 2y + 5 = 0$ .

eg3: Graph  $y^2 - 6x - 6y + 39 = 0$ . Include the focus, directrix, and axis of symmetry.



Do "The Parabola" #1-43 and/or p. 365 #1-11

## Identifying Conic Sections

General Form:

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

(where A and C are not both zero).

Condition on A and C	Conic Section	Example
$AC > 0, A = C$	CIRCLE	$4x^2 + 4y^2 + 2x + 3y + 1 = 0$
$AC > 0, A \neq C$	ELLIPSE	$4x^2 + 9y^2 + 2x + 3y + 1 = 0$
$A = 0 \text{ or } C = 0$	PARABOLA	$5x^2 + 2x + 3y - 1 = 0$
$AC < 0$	HYPERBOLA	$2x^2 - 4y^2 - 2x + 3y - 1 = 0$

e.g.: Determine the restrictions on General Form (above) for a parabola that opens to the left and has the  $x$ -axis as the axis of symmetry: ( $A, C, D, E$  only)

eg2: Determine the restrictions on General Form  
for an ellipse with its major axis on  
the y-axis: (A, C, and D only)

Do Qs # 5, 6, 43 on p.115-121  
and

# 10, 15, 16, 26-30, 37 on p. 122-125

in "Conics - Open-Ended Questions" handout  
(answers included)