

Ch. 1.1 - Arithmetic Sequences / Sequences

- a sequence is simply a list of numbers.
- there are two types:
 - i) FINITE - eg: 1, 4, 7, 10
 - ii) INFINITE - eg: 1, 4, 7, 10, 13, ...
- each number represents a term in the sequence.

eg: 1, 4, 7, 10, 13, ...
↑ ↑ ↑ etc...
term term term
1 2 3

- the term number represents the domain of the sequence (function), whereas the actual numbers in the list represent the range.

eg: 1, 4, 7, 10, 13, ...

Domain: $x = 1, 2, 3, 4, 5, \dots$

Range: $y = 1, 4, 7, 10, 13, \dots$

- thus, the domain of a sequence is a set of Natural numbers (positive integers).

- even though a sequence is a function, function notation is not used to represent a sequence; subscript notation is.

eg: $a_1, a_2, a_3, a_4, a_5, \dots$

- the subscript is the term of the sequence (domain).
- a_1, a_2, \dots represent the range.

$a_1 = 1^{\text{st}}$ term $a_3 = 3^{\text{rd}}$ term $a_n = n^{\text{th}}$ term

The entire sequence is denoted by $\{a_n\}$.

So...

A finite sequence possesses a domain which is a subset of natural numbers.

i.e. Domain of finite sequence: $x = \{1, 2, \dots, n\}$

An infinite sequence possesses a domain which is the entire (infinite) set of natural numbers.

i.e. Domain of infinite sequence: $x = \{1, 2, 3, \dots\}$

Note: the range of any sequence may include values ≤ 0 , along with rational numbers.

e.g. 1: Write the first four terms of each sequence:

a) $a_n = \frac{n+1}{n}$

$$a_1 = \frac{1+1}{1} = 2$$

$$a_2 = \frac{2+1}{2} = \frac{3}{2}$$

$$a_3 = \frac{3+1}{3} = \frac{4}{3}$$

$$a_4 = \frac{4+1}{4} = \frac{5}{4}$$

$$\left\{ 2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \dots \right\}$$

b) $b_n = 2n - 3$

$$b_1 = 2(1) - 3 = -1$$

$$b_2 = 2(2) - 3 = 1$$

$$b_3 = 2(3) - 3 = 3$$

$$b_4 = 2(4) - 3 = 5$$

$$\left\{ -1, 1, 3, 5, \dots \right\}$$

c) $t_n = 2^n$

$$t_1 = 2^1 = 2$$

$$t_2 = 2^2 = 4$$

$$t_3 = 2^3 = 8$$

$$t_4 = 2^4 = 16$$

$$\left\{ 2, 4, 8, 16, \dots \right\}$$

Recursive Sequence

- a sequence where each term is defined from earlier term(s) in the sequence. It requires knowledge of this term or terms.

eg2: Write the next four terms of the recursive formula: $a_1 = 3$ $a_n = \frac{a_{n-1}}{n}$

$$a_1 = 3 \quad a_2 = \frac{a_{2-1}}{2} = \frac{a_1}{2} = \frac{3}{2}$$

$$a_3 = \frac{a_{3-1}}{3} = \frac{a_2}{3} = \frac{\left(\frac{3}{2}\right)}{3} = \frac{1}{2}$$

$$a_4 = \frac{a_{4-1}}{4} = \frac{a_3}{4} = \frac{\left(\frac{1}{2}\right)}{4} = \frac{1}{8}$$

$$a_5 = \frac{a_{5-1}}{5} = \frac{a_4}{5} = \frac{\left(\frac{1}{8}\right)}{5} = \frac{1}{40}$$

$$\left\{ 3, \frac{3}{2}, \frac{1}{2}, \frac{1}{8}, \frac{1}{40}, \dots \right\}$$

Sigma Notation

- here and there, it is important to find the sum of a sequence.

$$\text{ie. } a_1 + a_2 + a_3 + \dots + a_n$$

- more often than not, sigma notation (utilizing the greek letter sigma (Σ)) is used to represent these situations.

$$\text{eg: } \underbrace{a_1 + a_2 + \dots + a_n}_{\text{expanded notation}} = \sum_{k=1}^n a_k$$

sigma notation

note: k is the index of the sum, showing where the summation begins. n shows where the summation ends.

another note: any summation possesses

$$\underline{n - (k-1) \text{ terms, or,}}$$

$$\underline{n - k + 1 \text{ terms.}}$$

eg3: Find the sum of each sequence:

a) $\sum_{k=1}^4 (2k+1)$ note, there are $4-1+1 = 4$ terms.

$$k=1: 2(1)+1=3, \quad k=2: 2(2)+1=5,$$

$$k=3: 2(3)+1=7, \quad k=4: 2(4)+1=9$$

$$3 + 5 + 7 + 9 = \boxed{24}$$

b) $\sum_{k=1}^5 (k^2+1)$ note, there are $5-1+1 = 5$ terms

$$k=1: (1)^2+1=2, \quad k=2: (2)^2+1=5$$

$$k=3: (3)^2+1=10, \quad k=4: (4)^2+1=17$$

$$k=5: (5)^2+1=26$$

$$2 + 5 + 10 + 17 + 26 = \boxed{60}$$

c) $\sum_{k=1}^3 (k^3-k)$ note, there are $3-1+1 = 3$ terms.

$$k=1: (1)^3-1=0, \quad k=2: (2)^3-2=6$$

$$k=3: (3)^3-3=24$$

$$0 + 6 + 24 = \boxed{30}$$

eg4: Re-write each summation in sigma notation:

a) $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \dots + \frac{12}{12+1}$

$$\begin{array}{l} n = 12 \\ k = 1 \end{array} \rightarrow \sum_{k=1}^{12} \frac{k}{k+1}$$

b) $\frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \dots + \left(\frac{2}{3}\right)^n$

$$\sum_{k=1}^n \left(\frac{2}{3}\right)^k$$

Arithmetic Sequence

- a sequence is deemed "Arithmetic" when the difference between successive terms is constant.

eg:
$$\boxed{2, 7, 12, 17, \dots}$$

$\underbrace{2}_{+5}, \underbrace{7}_{+5}, \underbrace{12}_{+5}, \underbrace{17}_{+5}, \dots$

This arithmetic sequence has a COMMON DIFFERENCE (d) of 5.
ie. $d = 5$

So:

$$1^{\text{st}} \text{ term } (a_1) = a_1$$

$$2^{\text{nd}} \text{ term } (a_2) = a_1 + d$$

$$\begin{aligned}3^{\text{rd}} \text{ term } (a_3) &= a_2 + d = (a_1 + d) + d \\&= a_1 + 2d\end{aligned}$$

$$\begin{aligned}4^{\text{th}} \text{ term } (a_4) &= a_3 + d = (a_1 + 2d) + d \\&= a_1 + 3d\end{aligned}$$

$$n^{\text{th}} \text{ term } (a_n) = a_1 + (n-1)d$$

for any integer $n \geq 1$

Again, for an arithmetic sequence $\{a_n\}$,
whose first term is a_1 , with common
difference d :

$$\underline{a_n = a_1 + (n-1)d}$$

eg5: What is d for $11, 8, 5, 2, \dots$?

$$\begin{aligned}11 + d &= 8 \\d &= -3\end{aligned}$$

$$\begin{aligned}8 + d &= 5 \\d &= -3\end{aligned}$$

etc...

$$\boxed{d = -3}$$

eg6: Determine if the sequence
 $\{t_n\} = \{3 - 2n\}$ is arithmetic.

$$\left. \begin{array}{l} t_1 = 3 - 2(1) = 1 \\ t_2 = 3 - 2(2) = -1 \\ t_3 = 3 - 2(3) = -3 \end{array} \right\} \begin{array}{l} d = -2 \\ d = -2 \\ d = -2 \end{array} \text{Yes!}$$

eg7: Find the 14th term of the arithmetic sequence 3, 7, 11, ...

$$\begin{array}{ll} a_1 = a = 3 & t_n = a + (n-1)d \\ d = 4 & \\ & t_{14} = 3 + (14-1)(4) \\ & = 3 + (13)(4) \\ & = \boxed{55} \end{array}$$

eg8: Which term in the arithmetic sequence 4, 7, 10, ... has a value of 439?

$$a_1 = a = 4 \quad t_n = a + (n-1)d$$

$$d = 3 \quad 439 = 4 + (n-1)(3)$$

$$439 = 4 + 3n - 3$$

$$438 = 3n \rightarrow \boxed{\begin{array}{l} n = 146 \\ * \text{the } 146^{\text{th}} \text{ term!} \end{array}}$$

eg 9: $t_7 = 78$; $t_{18} = 45$. The sequence is arithmetic. Find t_1 .

Find d :

$$t_n = t_1 + (n-1)d$$

$\swarrow t_7$

$\searrow t_{18}$

$$78 = t_1 + (7-1)d$$

$$45 = t_1 + (18-1)d$$

$$78 = t_1 + 6d$$

$$45 = t_1 + 17d$$

$$t_1 = \underbrace{78 - 6d}_{\text{curve}}$$

$$45 = (78 - 6d) + 17d$$

$$45 = 78 + 11d$$

$$11d = -33$$

$$d = -3$$

$$t_1 = 78 - 6d$$

$$= 78 - 6(-3)$$

$$= 78 + 18$$

$$= 96$$

$$\boxed{t_1 = 96}$$

y 10: Find x so that $3x+2$, $2x-3$, and $2-4x$ are consecutive terms in an arithmetic sequence.

SYSTEM OF EQUATIONS:

$$(3x+2) + d = 2x - 3$$

$$(3x+2) + 2d = 2 - 4x$$

$$(3x+2) + d = 2x - 3$$

$$(3x+2) + 2d = 2 - 4x$$

$$x + d = -5$$

$$7x + 2d = 0$$

$$d = -x - 5$$

$$2d = -7x$$

$$2(-x-5) = -7x$$

$$-2x - 10 = -7x$$

$$5x = 10$$

$$\boxed{x = 2}$$

p. 9-12

1-17 (18, 19 for 'fun')

Ch. 1.2 - Arithmetic Series

- a series is the sum of the terms of a sequence.

- a series can be finite or infinite



* this section *

Deriving the Sum Formula for Finite Arithmetic Series

Let $\{a_n\} = a_1, a_2, a_3, \dots, a_n$ be a finite arithmetic sequence.

Then $a_1 + a_2 + a_3 + \dots + a_n$ is a finite arithmetic series.

Let d = common difference

S_n = sum of the series

$$S_n = a_1 + a_2 + a_3 + \dots + a_n$$

$$\text{eq'n 1: } S_n = a + (a+d) + (a+2d) + \dots + (a+(n-1)d)$$

Let $l = a + (n-1)d \rightarrow$ the last term

$$\text{eq'n 2: } S_n = l + (l-d) + (l-2d) + \dots + a \quad (\text{SUM IN REVERSE ORDER})$$

Add eq'n 1 and 2:

$$S_n = a + (a+d) + (a+2d) + \dots + l$$

$$\textcircled{+} \quad S_n = l + (l-d) + (l-2d) + \dots + a$$

$$2S_n = (a+l) + (a+l) + (a+l) + \dots + (a+l)$$

$$2S_n = n(a + l)$$

$$S_n = \frac{n}{2}(a + l)$$

$$S_n = \frac{n}{2}(a + (a + (n-1)d))$$

$$S_n = \frac{n}{2}(2a + (n-1)d)$$

SUM OF A FINITE ARITHMETIC SERIES:

The sum of the first n terms of an arithmetic series is given by:

$$S_n = \frac{n}{2}(a + l)$$

OR

$$S_n = \frac{n}{2}(2a + (n-1)d)$$

where $a = a_1$ = the 1st term

$l = a + (n-1)d$ = the last (n^{th}) term

d = common difference

eg1: Find the sum of the even integers from 2 to 60 inclusive.

$$n = \frac{60}{2} = 30$$

$$a = 2 \quad l = 60 \quad d = 2$$

$$S_n = \frac{n}{2} (a + l)$$

$$S_{30} = \frac{30}{2} (2 + 60)$$

$$S_{30} = 15(62) = \boxed{930}$$

eg2: Find the sum of the first 25 terms of the series $11 + 15 + 19 + \dots$

$$a = 11 \quad d = 4 \quad n = 25$$

$$S_n = \frac{n}{2} (2a + (n-1)d)$$

$$S_{25} = \frac{25}{2} (2(11) + (25-1)(4))$$

$$S_{25} = \frac{25}{2} (22 + 96)$$

$$S_{25} = \frac{25}{2} (118)$$

$$S_{25} = \boxed{1475}$$

eg3: Find the sum of the series

$$7 + 10 + 13 + \dots + 100$$

$$\begin{array}{ll} a = 7 & l = 100 \\ d = 3 & \end{array}$$

$$\text{Find } n: \quad l = a + (n-1)d$$

$$100 = 7 + (n-1)(3)$$

$$100 = 7 + 3n - 3$$

$$96 = 3n$$

$$n = 32$$

$$S_n = \frac{n}{2} (a + l)$$

$$S_{32} = \frac{32}{2} (7 + 100)$$

$$S_{32} = \boxed{1712}$$

Summation (Sigma Notation)

- if the summation expression is LINEAR,
then the summation is an arithmetic series.

eg: $\sum_{k=1}^{10} (2k+1) \rightarrow \underline{\text{arithmetic}}$, since $2k+1$ is linear

$\sum_{k=1}^{10} (k^2+1) \rightarrow \underline{\text{not arithmetic}}$, since k^2+1 is not linear.

eg4: Evaluate $\sum_{k=1}^{100} (2k + 1)$

$$a = 2(1) + 1 = 3 \quad d = 2$$

$$l = 2(100) + 1 = 201 \quad n = 100 + 1 - 1 = 100$$

$$S_n = \frac{n}{2} (a + l)$$

$$S_{100} = \frac{100}{2} (3 + 201)$$

$$S_{100} = \boxed{10200}$$

eg5: Write $5 + 9 + 13 + \dots + 137$ in summation (sigma) notation.

Find n :

$$5 = dk + b$$

$$l = a + (n-1)d$$

$$5 = 4(1) + b$$

$$137 = 5 + (n-1)(4)$$

$$\underline{b = 1}$$

$$137 = 5 + 4n - 4$$

$$9 = dk + b$$

$$4n = 136$$

$$9 = 4(2) + b$$

$$n = 34$$

$$b = 1$$

$$\boxed{\sum_{k=1}^{34} 4k + 1}$$

eg6: The sum of the first n terms of an arithmetic series is $S_n = 5n^2 - 3n$.
Find d .

$$S_1 = 5(1)^2 - 3(1) = 2$$

$$S_2 = 5(2)^2 - 3(2) = 14$$

$$a = 2$$

$$S_2 = a + a_2$$

$$14 = 2 + a_2$$

$$a_2 = 12$$

$$d = a_2 - a = 12 - 2 = \boxed{10}$$

eg7: Find the sum of all multiples of 6 between 100 and 1000.

$$\left. \begin{array}{l} \text{First multiple? } 102 \\ \text{Last multiple? } 996 \end{array} \right\} 102 + \dots + 996$$

$$l = a + (n-1)d$$

$$996 = 102 + (n-1)6$$

$$996 = 102 + 6n - 6$$

$$900 = 6n$$

$$n = 150$$

$$S_n = \frac{n}{2}(a + l)$$

$$S_{150} = \frac{150}{2}(102 + 996)$$

$$S_{150} = \boxed{82350}$$

~~eg8:~~ Find two arithmetic means between
8 and 29.

i.e. $8 \underbrace{\quad}_{+d} \underbrace{\quad}_{+d} \underbrace{\quad}_{+d} 29$

$$8 + 3d = 29$$

$$3d = 21$$

$$d = 7$$

15 and 22

p. 16-19

1-6, 8-12 * try 7, 13, 14 for fun.

Ch. 1.3 - Geometric Sequences

- recall that arithmetic sequences possess terms that differ by a common difference, d . (ie. a constant number is ADDED to each term to get the next term).
- in a GEOMETRIC sequence, each term is MULTIPLIED by a constant number to get the next term. (ie. geometric sequences possess terms that differ by a common ratio (or factor of), r .

eg: 2, 4, 8, 16, 32, etc...

$$2 \times 2 = 4 \quad 4 \times 2 = 8 \quad 8 \times 2 = 16, \text{ etc.}$$

$$r = 2.$$

Geometric Sequence

A sequence is GEOMETRIC if the ratio of consecutive terms is constant.

i.e. $a_1, a_2, a_3, \dots, a_n$ is geometric if there is a number r ($r \neq 0$), such that:

$$\frac{a_2}{a_1} = r, \quad \frac{a_3}{a_2} = r, \quad \frac{a_n}{a_{n-1}} = r.$$

again,
 $r = \text{COMMON RATIO}$

e.g. 1: For each geometric sequence, find the common ratio:

a) 2, 6, 18, 54, ...

$$\begin{array}{l} 2r = 6 \\ r = 3 \end{array}$$
 or $\frac{6}{2} = r = 3$

$$\boxed{\begin{array}{l} 6r = 18 \\ r = 3 \end{array}}$$

b) 3, -6, 12, -24, ...

$$\begin{array}{l} 3r = -6 \\ r = -2 \end{array}$$

$$\begin{array}{l} -6r = 12 \\ r = -2 \end{array}$$

$$\boxed{r = -2}$$

c) -8, -4, -2, -1, ...

$$\begin{array}{l} -8r = -4 \\ r = \frac{1}{2} \end{array}$$

$$\begin{array}{l} -4r = -2 \\ r = \frac{1}{2} \end{array}$$

$$\boxed{r = \frac{1}{2}}$$

The n^{th} term of a Geometric Sequence:

$$t_n = ar^{n-1}, \text{ for any integer } n \geq 1.$$

a = first term, t_1

Derivation:

Let $a = a_1$, = the first term

Let r = common ratio.

$$a_1 = a$$

$$a_2 = ar$$

$$a_3 = a_2 r = ar(r) = ar^2$$

$$a_4 = a_3 r = ar^2(r) = ar^3$$

⋮

$$a_n = ar^{n-1}$$

* another view: the exponent in each term
is one less than the subscript of the term.

* also: $a_m = a_n r^{m-n}$

e.g. 2: Find the 8th term of the geometric sequence: 3, 12, 48, 192, ...

$$3r = 12 \quad a = 3$$

$$r = 4$$

$$\begin{aligned} a_n &= ar^{n-1} \\ a_8 &= 3(4)^{8-1} \end{aligned} \quad \rightarrow \quad \boxed{\begin{aligned} a_8 &= 3(4)^7 \\ a_8 &= 49152 \end{aligned}}$$

Eg3: The 4th term of a geometric sequence is 125. The 9th term of the same sequence is $\frac{125}{32}$. Find the 13th term.

$$t_4 = ar^3 = 125$$

$$t_9 = ar^8 = \frac{125}{32}$$

$$(ar^3)(x) = ar^8 \quad (\text{ie. } 125z = \frac{125}{32})$$

$$r^3 x = r^8$$

$$x = \frac{r^8}{r^3} = r^5 \quad (\text{ie. } z = r^5)$$

$$\text{so, } ar^3 \cdot r^5 = ar^8$$

$$125 r^5 = \frac{125}{32}$$

$$r^5 = \frac{\left(\frac{125}{32}\right)}{125}$$

$$r^5 = \frac{1}{32}$$

$$\sqrt[5]{r^5} = \sqrt[5]{\frac{1}{32}}$$

$$r = \frac{1}{2}$$

$$ar^3 = 125$$

$$a\left(\frac{1}{2}\right)^3 = 125$$

$$a = \frac{125}{\left(\frac{1}{8}\right)}$$

$$a = 1000$$

$$t_{13} = ar^{12} = 1000 \left(\frac{1}{2}\right)^{12} = \frac{1000}{4096} = \boxed{\frac{125}{512}}$$

eg 4: If $x, 2x+2, 3x+3, \dots$ represents a geometric sequence, then what is the value of x ? What is r ?

$$xr = 2x + 2$$

$$(2x+2)r = 3x+3$$

$$r = \frac{2x+2}{x}$$

$$r = \frac{3x+3}{2x+2}$$

$$\frac{2x+2}{x} = \frac{3x+3}{2x+2}$$

note: $x \neq 0, -1$

$$(2x+2)^2 = x(3x+3)$$

$$4x^2 + 8x + 4 = 3x^2 + 3x$$

$$x^2 + 5x + 4 = 0$$

$$(x+4)(x+1) = 0$$

$$\boxed{x = -4, \cancel{x=1}}$$

$$r = \frac{2x+2}{x} = \frac{2(-4)+2}{-4} = \frac{-6}{-4} = \frac{3}{2}$$

or

$$r = \frac{2x+2}{x} = \frac{2(-1)+2}{-1} = 0 \quad r \text{ cannot be } 0, \\ \text{so } x \neq -1$$

p. 22-25 #1-5, 7-19. (*6, 20 for fun)

NOTE: #17 should read: "after 12 SUCCESSIVE swings." *15 → after 9 cycles.

Ch. 1.4 - Geometric Series

A geometric series is the sum of all of the terms of a geometric sequence.

Deriving the Sum Formula for Finite Geometric Series

Let $S_n = a + ar + ar^2 + ar^3 + \dots + ar^{n-1}$ be a geometric series with common ratio, r .

S_n can be re-expressed by multiplying S_n by $-r$, then adding the result to S_n :

$$-rS_n = (-r)(a) + (-r)(ar) + (-r)(ar^2) + \dots + (-r)(ar^{n-1})$$

$$-rS_n = -ar - ar^2 - ar^3 - \dots - ar^{n-1} - ar^n$$

$$+ S_n = a + ar + ar^2 + ar^3 + \dots + ar^{n-1}$$

$$S_n - rS_n = a - ar^n$$

$$\text{Note: } t_n = ar^{n-1} = l$$

$$S_n(1-r) = a - ar^n$$

$$S_n = \frac{a - ar^n}{1-r}$$

OR

$$S_n = \frac{a - ar^n}{1-r}$$

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_n = \frac{a - ar^{n-1} \cdot r}{1-r}$$

$$S_n = \frac{a - rl}{1-r}$$

The sum of the first n terms of a geometric series with first term, a , and last term, l (or ar^{n-1}), is given by:

$$S_n = \frac{a(1-r^n)}{1-r}$$

or

$$S_n = \frac{a - ar^n}{1 - r}$$

note: $r \neq 1$

or

$$S_n = \frac{a - rl}{1 - r}$$

Eg1: Find the sum of the geometric series
 $2 + 6 + 18 + 54 + \dots 1458.$

$$a = 2$$

$$r = \frac{6}{2} = 3$$

$$l = 1458$$

$$\begin{aligned} S_n &= \frac{a - rl}{1 - r} = \frac{2 - (3)(1458)}{1 - 3} \\ &= 2186 \end{aligned}$$

Eg2: Find the sum of the first eight terms of the geometric series $3 + 6 + 12 + \dots$

$$a = 3 \quad r = \frac{6}{3} = 2$$

$$n = 8$$

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

$$S_8 = \frac{3(1 - 2^8)}{1 - 2} = \boxed{765}$$

Eg3: Find the sum of the geometric series

$$\sum_{k=1}^{10} 3(-2)^{k-1}$$

$$\text{find } a: k=1 \rightarrow 3(-2)^{1-1} = 3 = a$$

$$\text{find } l: k=10 \rightarrow 3(-2)^{10-1} = 3(-2)^9 = -1536 = l$$

$$r = -2 \rightarrow \text{check w/ } k=2 \\ \hookrightarrow t_2 = -6$$

$$S_n = \frac{a - rl}{1 - r} \rightarrow S_{10} = \frac{3 - (-2)(-1536)}{1 - (-2)}$$

$$\boxed{S_{10} = -1023}$$

Note: in sigma (summation) notation,
 an ARITHMETIC series has its variable
 in the base. A GEOMETRIC series has
 its variable in the exponent.

eg: ARITHMETIC : $\sum_{k=1}^5 (2k - 3)$ $d = \underline{2}$

GEOMETRIC : $\sum_{k=1}^5 2 \cdot 3^k$ $r = \underline{3}$

NEITHER : $\sum_{k=1}^5 (2 + 3^k)$

Why? $k=1 \rightarrow \frac{5}{1}$ $k=3 \rightarrow \frac{29}{27}$
 $k=2 \rightarrow \frac{11}{9}$ $k=4 \rightarrow \frac{83}{81}$
 no trend.

q4: Write the geometric series, $6 + 18 + 54 + 162 + 486$, using sigma notation with index $k = 1$.

FIVE terms : $n = 5$ $\frac{18}{6} = 3 = r$

$$\sum_{k=1}^5 3^k \rightarrow 3^1 \cdot ? = 6 \quad ? = 2$$

$$\sum_{k=1}^5 2 \cdot 3^k$$

p.28-31
 #1-19 (omit 14, 20)
 typo #17 answer: $t_2 = -8$

typo #12 answer: using S_n : $a = \underline{20177.28}$

Ch. 1.5 - Infinite Geometric Series

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \dots, \left(\frac{1}{2}\right)^{n-1} \quad n \in \{\text{NATURAL numbers}\}$$

an example of an **INFINITE** geometric sequence.

- as n gets larger, $\left(\frac{1}{2}\right)^{n-1}$ gets closer and closer to 0.

$$\begin{aligned} \text{eg: } n = 51 &\rightarrow \left(\frac{1}{2}\right)^{51-1} = \left(\frac{1}{2}\right)^{50} = 8.8 \times 10^{-16} \\ n = 101 &\rightarrow \left(\frac{1}{2}\right)^{101-1} = \left(\frac{1}{2}\right)^{100} = 7.88 \times 10^{-31} \end{aligned}$$

ie. $\lim_{n \rightarrow \infty} \left(\frac{1}{2}\right)^n = 0 \rightarrow$ as n approaches an infinitely large number, $\left(\frac{1}{2}\right)^n$ approaches 0.

- this is an example of a CONVERGENT geometric sequence.

ie. as the terms of an infinite geometric sequence get closer and closer to some real number, the sequence is said to be CONVERGENT.

$$1, 2, 4, 8, 16, 32, \dots, 2^{n-1} \quad n \in \{\text{NATURAL numbers}\}$$

- this sequence does NOT converge to some real number.

- a sequence that does not converge is said to be a DIVERGENT geometric sequence.

Convergent geometric sequences possess an r-value between -1 and 1 ($-1 < r < 1$) or $|r| < 1$.

Divergent geometric sequences possess an r-value less than/equal to -1 or greater than/equal to 1 ($r \leq -1$ or $r \geq 1$).

The Sum of an Infinite Geometric Sequence

is given by:

$$S_{\infty} = \frac{a}{1-r}, \quad |r| < 1 \\ (-1 < r < 1)$$

Proof :

$$S_n = \frac{a(1-r^n)}{1-r}$$

as $n \rightarrow \infty$, $r^n \rightarrow 0$ (when $|r| < 1$)

$$S_n = S_\infty = \frac{a(1-0)}{1-r}$$

$$S_\infty = \frac{a}{1-r}$$

e.g.: Find the sum of each infinite geometric series:

a) $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \dots$

$$S_\infty = \frac{a}{1-r}$$

$$\frac{1}{3}r = \frac{1}{9}$$

$$r = \frac{1}{3}$$

$$a = \frac{1}{3}$$

$$S_\infty = \frac{\left(\frac{1}{3}\right)}{1-\frac{1}{3}} = \frac{\left(\frac{1}{3}\right)}{\left(\frac{2}{3}\right)} = \boxed{\frac{1}{2}}$$

$$b) \frac{2}{5} - \frac{4}{15} + \frac{8}{45} - \dots$$

$$\frac{2}{5}r = -\frac{4}{15}$$

$$S_{\infty} = \frac{a}{1-r}$$

$$30r = -20$$

$$r = -\frac{2}{3}$$

$$a = \frac{2}{5}$$

$$S_{\infty} = \frac{\left(\frac{2}{5}\right)}{1 - \left(-\frac{2}{3}\right)} = \frac{\left(\frac{2}{5}\right)}{\left(\frac{5}{3}\right)} = \boxed{\frac{6}{25}}$$

SUMMATION (SIGMA) FORM

An infinite geometric series with first term a and common ratio r ($|r| < 1$) is

given by :

$$\sum_{i=1}^{\infty} ar^{i-1}$$

eg2: Find the sum of $\sum_{k=2}^{\infty} 8\left(-\frac{1}{2}\right)^{k-1}$

$$a = 8\left(-\frac{1}{2}\right)^{2-1} = 8\left(-\frac{1}{2}\right) = -4$$

$$r = -\frac{1}{2}$$

$$S_{\infty} = \frac{a}{1-r} = \frac{-4}{1 - \left(-\frac{1}{2}\right)} = \frac{-4}{\left(\frac{3}{2}\right)} = \boxed{-\frac{8}{3}}$$

q3: Initially, a pendulum swings through an arc of 25 cm. On each successive swing, friction lessens the length of the pendulum swing by 5%. When the pendulum eventually stops, what total length will it have swung?

$$* \text{ note: } 100\% - 5\% = 95\%$$

each successive swing is 95% the length of the previous swing.

$$r = 0.95$$

$$S_{\infty} = \frac{a}{1-r} = \frac{25}{1-0.95} = \boxed{500 \text{ cm}}$$

q4: A ball is dropped from 12 ft. and rebounds $\frac{2}{3}$ the distance from which it fell. Find the total distance the ball traveled.

$$12 \cdot \frac{2}{3} = 8 \text{ ft.} * \text{ first bounce up is 8 ft.}$$

$$a = \frac{8}{(\text{up})} + \frac{8}{(\text{down})} = 16$$



$$S_{\infty} = \frac{a}{1-r} + 12 = \frac{16}{1-\left(\frac{2}{3}\right)} + 12$$

$$= 48 + 12 \\ = \boxed{60 \text{ ft.}}$$

q5: Write $0.\overline{24}$ as a fraction.

$$0.\overline{24} = 0.2 + 0.\overline{04}$$

$$= \frac{2}{10} + \frac{4}{100} + \frac{4}{1000} + \frac{4}{10000} + \dots$$

$$\left. \begin{array}{l} \frac{2}{10} r = \frac{4}{100} \\ r = \frac{2}{10} = \frac{1}{5} \end{array} \right\} \text{BUT! } \frac{4}{100} \left(\frac{1}{5} \right) = \frac{4}{500} \text{ not } \frac{4}{1000}$$

$$= \frac{2}{10} + \underbrace{\frac{4}{100} + \frac{4}{1000} + \frac{4}{10000}}_{r = \frac{1}{10}} + \dots$$

$$S_\infty = \frac{a}{1-r} + \frac{2}{10}$$

$$= \frac{\left(\frac{4}{100}\right)}{\left(1 - \frac{1}{10}\right)} + \frac{1}{5} = \frac{\left(\frac{4}{100}\right)}{\left(\frac{9}{10}\right)} + \frac{1}{5} = \frac{4}{100} \cdot \frac{10}{9} + \frac{1}{5} = \frac{4}{90} + \frac{1}{5} = \frac{2}{45} + \frac{1}{5}$$

$$= \frac{2}{45} + \frac{9}{45}$$

$$= \boxed{\frac{11}{45}}$$

eg6: Solve for x :

$$\sum_{k=1}^{\infty} (\tan x)^{k-1} = 1 \quad 0^\circ \leq x < 45^\circ$$

$$a = (\tan x)^{1-1} = (\tan x)^0 = 1$$

$$r = \tan x$$

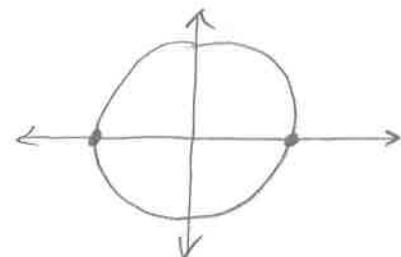
$$S_{\infty} = \frac{a}{1-r} = 1$$

$$\frac{1}{1-\tan x} = 1$$

$$1 - \tan x = 1$$

$$\tan x = 0$$

$$\boxed{x = 0^\circ}, \cancel{180^\circ}$$



$$\tan x = \frac{y}{x} = \frac{0}{\pm 1}$$

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1-5, 7-20.

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Chapter Review