

Ch. 7.1 - Fundamental Counting Principle

Ex1: A cafe has a lunch special consisting of an egg or ham sandwich (E or H), milk(M), juice(J), or coffee (C) and yogourt (Y) or pie (P) for dessert. If one item is chosen from each category, how many different meals are possible?

EMY	EMP	HMY	HMP	} twelve possible meals
EJY	EJP	HJY	HJP	
ECY	ECP	HCY	HCP	

$$2 \times 3 \times 2 = \underline{\underline{12}}$$

Sandwich choices drink choices dessert choices

We say that the acts of choosing a sandwich, choosing a drink, then choosing a dessert are INDEPENDENT EVENTS, since the outcome of no one event influences another's.

FUNDAMENTAL COUNTING PRINCIPLE -

If one item can be selected in a different ways, a second item in b different ways, a third item in c different ways, and so on, then... the total number of ways to select one of each item is: $a \cdot b \cdot c \dots$

Eg 2: A computer retailer sells 5 different computers, 3 different monitors, 5 different printers, and 2 different multimedia packages. How many different computer system combo's are available?

$$5 \times 3 \times 5 \times 2 = \boxed{150}$$

Eg 3: How many different two-digit numbers are there?

choices: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9

ten digits

but, 1st digit cannot be zero.

$$9 \times 10 = \boxed{90}$$

Eg 4: How many EVEN two-digit numbers exist?

First digit cannot be zero: 9 choices

2nd digit EVEN: 0, 2, 4, 6, 8 → 5 choices

$$9 \times 5 = \boxed{45}$$

eg5: How many two-digit numbers can be formed using only the digits 0, 1, 3, 5, 7, 9?

a) Repetitions allowed:

$$5 \times 6 = \boxed{30}$$

b) Repetitions not allowed:

$$5 \times 5 = \boxed{25}$$

eg6: A T/F test has seven questions. How many different possible answer sequences exist for the test?

$$2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \\ = 2^7 = \boxed{128}$$

eg7: A M/C exam has twenty questions with four possible answers to each question. How many different possible answer sequences are there for these twenty questions?

$$4 \times 4 \times \dots \text{ (20 times!)} = 4^{20} = 1.1 \times 10^{12}$$

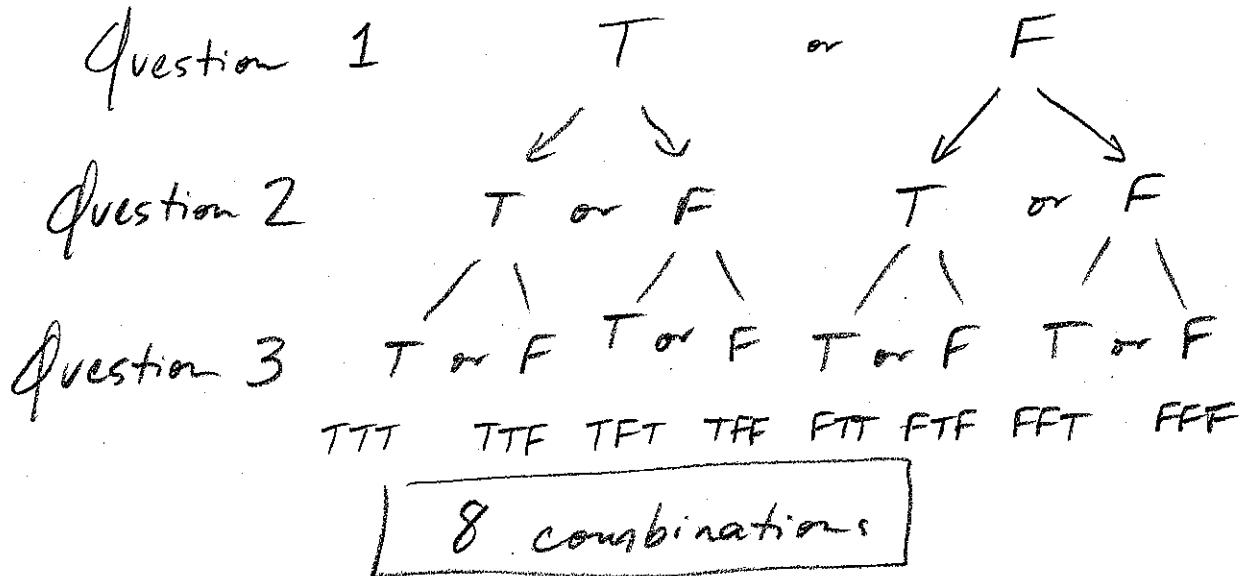
$$\boxed{110000000000}$$

1.1 trillion

Tree Diagrams

- useful way of visualizing and counting the number of outcomes of an event.

~~eg:~~ How many ways can a three-item (question) T/F test be answered?



* easier with the Fundamental Counting

Principle: $2 \times 2 \times 2 = 2^3 = \boxed{8}$

~~eg:~~ How many different ways can five different books be arranged on a shelf?

* 5 possibilities for 1st pick;

4 for next;

then 3, etc...

$$5 \times 4 \times 3 \times 2 \times 1 = \boxed{120}$$

The product of consecutive integers from n to 1 is given by $n!$ (n factorial).

Definition of $n!$:

let n be a positive integer,

$$n! = \boxed{n(n-1)(n-2)\dots(3)(2)(1)}.$$

Note: $0! = 1$ by definition (to be proven later).

9/10: Simplify each of the following:

a) $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = \boxed{720}$

b) $\frac{99!}{96!} = \frac{99 \cdot 98 \cdot 97 \cdot 96!}{96!} = \boxed{941094}$

c) $\frac{15! - 13!}{13!} = \frac{15!}{13!} - \frac{13!}{13!} = \frac{15 \cdot 14 \cdot 13!}{13!} - 1$
 $= 210 - 1 = \boxed{209}$

d) $\frac{(n+2)!}{(n-1)!} = \frac{(n+2)(n+1)(n)(n-1)!}{(n-1)!} = \boxed{n^3 + 3n^2 + 2n}$

$$\begin{aligned}
 e) \frac{n!}{(n-2)! + (n-3)!} &= \frac{(n)(n-1)(n-2)(n-3)!}{(n-2)(n-3)! + (n-3)!} \\
 &= \frac{(n)(n-1)(n-2)(n-3)!}{(n-3)! ((n-2)+1)} \\
 &= \frac{n(n-1)(n-2)}{(n-1)} \\
 &= n(n-2) = \boxed{n^2 - 2n}
 \end{aligned}$$

eq 11: Solve for n :

$$\frac{n!}{(n-2)!} = 12$$

$$\frac{(n)(n-1)(n-2)!}{(n-2)!} = 12$$

$$n^2 - n = 12$$

$$n^2 - n - 12 = 0$$

$$(n-4)(n+3) = 0$$

$$\boxed{n=4}, \cancel{n=-3}$$

n must
be positive!

Hwk: Read eq 8 p.318

Do # 1-12, 16, 17, ~~18, 19.~~
(#4 needs tree diagram)
(#9c,d, 10 'tricky')

Also: Handout 6.1 exercises: #8, 9, 11qcd
14, 15 a, 17,

Ch. 7.2 - Permutations

eg1: How many different arrangements can be made using the letters A, B, and C once only?

ABC	ACB	}	6
BAC	BCA		
CBA	CAB		

using Fundamental Counting Principle:

$$\boxed{3} \times \boxed{2} \times \boxed{1} = \underline{\underline{6}}$$

A PERMUTATION is an arrangement of a set of n different objects. In a permutation, ORDER is important and no object is repeated.

eg2: How many permutations are there of the five letters A, B, C, D, and E?

$$\underbrace{5 \times 4 \times 3 \times 2 \times 1}_{\text{5 positions}} = 5! = \boxed{120}$$

So... for n different objects, there are:

$$\underbrace{n(n-1)(n-2)\dots(3)(2)(1)}_{\text{n positions}} = n! \text{ different permutations.}$$

Q3:

- a) How many different ways can ten different books be arranged on a shelf?

$$10! = \boxed{3628800}$$

- b) How many different ways can four books be arranged on a shelf if they are selected from ten different books?

$$10 \times 9 \times 8 \times 7 = \boxed{5040}$$

Let's look closer:

$$10 \times 9 \times 8 \times 7 = \frac{10!}{6!} = \frac{10!}{(10-4)!}$$

So: PERMUTATION OF n OBJECTS TAKEN r AT A TIME:

The number of permutations of n different objects taken (r at) a time is given by:

$$\left[\frac{n!}{(n-r)!} = nPr = P(n, r) \right]$$

eg4: How many three-letter permutations can be formed using all letters in the word CLARINET?

→ eight letters, so...

$$8P_3 = \frac{8!}{(8-3)!} = \frac{8!}{5!} = \frac{8 \cdot 7 \cdot 6}{\boxed{336}}$$

eg5: How many ways can a president, vice-president, and treasurer be selected from a class of thirty students?

$$30P_3 = \frac{30!}{27!} = 30 \cdot 29 \cdot 28 = \boxed{24360}$$

Proof that $0! = 1$:

The number of ways to arrange all n different objects is given by:

$$nP_n \text{ and } n!$$

$$\text{So, } nP_n = n!$$

$$\frac{n!}{(n-n)!} = n!$$

$$(n-n)! = \boxed{0! = 1} \quad \text{PROOF!}$$

* up to this point, all permutations have been distinguishable, meaning that all objects have been different from one another.
(ie. no repetitions allowed).

Eg: The number of permutations of the four letters a, b, c, d is:

$$4P_4 = 4! = \underline{\underline{24}}$$

BUT, what if the four letters were a, a, b, c?

- you would have permutations that are non-distinguishable from one another.

(eg: $\overbrace{a, b, c, a}^4$ and a, b, c, a are the same)

Thus, half of the arrangements would be the same.

PERMUTATIONS WITH REPETITION

The number of permutations of n objects in which there are a alike of one kind, b alike of another, and so on, is given by:

$$\frac{n!}{a! b! c!}$$

Question above?

$$\frac{4!}{2!} = \frac{24}{2} = \boxed{12}$$

eg 6: How many permutations are there of the letters in the word MISSISSIPPI?

$$11 \overset{\rightarrow}{\text{letters}} = n$$

4 Ss, 4 Is, 2 Ps

$$\frac{11!}{4! 4! 2!} = \frac{11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5}{4! 2!} = 34650$$

eg 7: How many different ways can you arrange twelve flags in a row if five are red, four are blue, and three are yellow?

$$5 + 4 + 3 = 12 = n$$

$$\frac{12!}{5! 4! 3!} = 27720$$

eg 8: A T/F test has seven questions. How many answer keys are possible if three are true and four are false?

$$n = 7$$

$$\frac{7!}{4! 3!} = \underline{\underline{35}}$$

FyI:
Words that indicate
a PERMUTATION:

arrange
assign
order
code

Hwk: p. 326 * 1-3, 5, 7-13, 15, 16, 18-20 + Handout
Part II

Ch. 7.3 - Combinations

- the distinguishing aspect of a permutation is that the order of the objects is significant (eg: ab and ba are different).
- when order does not matter, we do not have an 'arrangement' of objects, but rather a 'collection' of objects.
- we say that a COMBINATION is an un-ordered collection of n different objects.

Examples of Combinations :

- five-card hand (poker)
- four-topping pizza
- three-person committee (where all people have equal power \Rightarrow unlike P, VP, treasurer example)

Combination Formula:

The number of combinations of n different objects taken r at a time is given by:

$$nCr = C(n, r) = \begin{bmatrix} n \\ r \end{bmatrix} = \frac{n!}{r!(n-r)!} = \frac{nPr}{r!}$$

e.g. a) How many different committees of three people can be formed from a group of seven if the first serves as the Chairperson, the second as the Treasurer, and the third as Secretary?

$$7P_3 = \frac{7!}{4!} = \boxed{210}$$

b) How many different committees of three people can be selected from a group of seven people?

$$7C_3 = \left[\frac{7}{3} \right] = \frac{7P_3}{3!} = \frac{210}{6} = \boxed{35}$$

Q2: A standard deck of 52 cards consists of four suits (spades, clubs, hearts, diamonds), with 13 cards in each suit.

a) How many five-card hands can be formed?

* keep in mind: order does not matter here

→ A ♠ K ♥ Q ♦ 4 ♦ 2 ♥

Same as

Q ♦ 4 ♦ K ♥ 2 ♥ A ♠

$$52C_5 = \left[\frac{52}{5} \right] = \frac{52P_5}{5!} = \frac{\left(\frac{52!}{47!} \right)}{5!} = \boxed{2598960}$$

(not $52P_5$)

b) How many five-card hands can be formed that:

b) consist of all hearts?

* 13 hearts
in deck

$$13C_5 = \left[\begin{matrix} 13 \\ 5 \end{matrix} \right] = \frac{\left(\frac{13!}{8!} \right)}{5!} = \boxed{1287}$$

c) consist of all face cards?

* 12 face cards
in deck
(three per suit)

$$12C_5 = \left[\begin{matrix} 12 \\ 5 \end{matrix} \right] = \frac{\left(\frac{12!}{7!} \right)}{5!} = \boxed{792}$$

d) consist of three hearts and two spades?

13 in deck
(choose 3)

INDEPENDENT
EVENTS

use Fundamental Counting
Principle.

13 in deck
(choose 2)

$$13C_3 \times 13C_2 = \left[\begin{matrix} 13 \\ 3 \end{matrix} \right] \cdot \left[\begin{matrix} 13 \\ 2 \end{matrix} \right] = \frac{(13!)}{3!} \cdot \frac{(13!)}{2!}$$

$$= 286 \cdot 78 \\ = \boxed{22308}$$

e) consist of exactly three hearts?

* two events \rightarrow 3 hearts + 2 non-hearts \rightarrow 39 in deck
(since $3 < 5$) (choose 2)

total cards in hand

$$13C_3 \cdot 39C_2 = \frac{\left(\frac{13!}{10!} \right)}{3!} \cdot \frac{\left(\frac{39!}{37!} \right)}{2!} = 286 \cdot 741 \\ = \boxed{211926}$$

f) consist of exactly three aces?

again, two events:

4 aces in deck + 48 non-aces
(choose 3) (choose 2)

$$4C_3 \cdot 48C_2 = \frac{(4!)}{3!} \cdot \frac{(48!)}{46!} = 4 \cdot 1128$$

= 4512

g) consist of at least (three or more) hearts?

- * new form of question
- must consider 3-heart hands, 4-heart hands, and 5-heart hands

add combos together

$$\text{three-heart} = 13C_3 \cdot 39C_2 = 286 \cdot 741 = 211926$$

$$+$$
 + + +

$$\text{four-heart} = 13C_4 \cdot 39C_1 = 715 \cdot 39 = 27885$$

$$+$$
 + + +

$$\text{five-heart} = 13C_5 = 1287 = 1287$$

241098

h) consist of, at most, two face cards?
(so, 2 or less)

$$\begin{aligned}
 &= 2 \text{ face hands} + 1 \text{ face hands} + 0 \text{ face hands} \\
 &= \binom{12}{2} \cdot \binom{40}{3} + \binom{12}{1} \cdot \binom{40}{4} + \binom{40}{5} \\
 &= (66 \cdot 9880) + (12 \cdot 91390) + 658008 \\
 &= \boxed{2406768}
 \end{aligned}$$

eg 3: A bag contains five red and four white balls. In how many ways can four balls be chosen such that exactly 2 are red?

$$\begin{matrix}
 5 \cdot C_2 & \times & C_2 \\
 \text{RED} & & \text{WHITE} \\
 & & (\text{non-red})
 \end{matrix} = \boxed{60}$$

eg 4: Solve for n given $nC_2 = 10$

$$\begin{aligned}
 \begin{bmatrix} n \\ 2 \end{bmatrix} &= 10 & \frac{(n)(n-1)(n-2)!}{(n-2)!} &= 20 \\
 \frac{\left(\frac{n!}{(n-2)!}\right)}{2!} &= 10 & n^2 - n &= 20 \\
 \frac{n!}{(n-2)!} &= 20 & n^2 - n - 20 &= 0 \\
 && (n-5)(n+4) &= 0 \\
 && \boxed{n=5}, & \cancel{*} \text{reject}
 \end{aligned}$$

eg5: If night school offers 100 courses, eight of which are in Math, and you select four courses by random selection, how many possibilities would include exactly one Math course?

$${}_{18}C_1 \cdot {}_{92}C_3 = \boxed{1\ 004\ 640}$$

Read p.330 examples 3 and 6

Do Qs 1-11^(a-d) p.331-333.

+
Handout Part III

Ch. 7.4 - Binomial Theorem

The BINOMIAL THEOREM is a method for expanding an expression of the form $(x+y)^n$, where n is a positive integer.

$$(x+y)^1 = x+y$$

$$(x+y)^2 = x^2 + 2xy + y^2$$

$$\begin{aligned}(x+y)^3 &= (x+y)^1(x+y)^2 = (x+y)(x^2 + 2xy + y^2) \\ &= x^3 + 3x^2y + 3xy^2 + y^3\end{aligned}$$

- as the exponent gets larger, this process gets more and more onerous.
 - useful to find a way of expanding a binomial without having to multiply repeatedly

PASCAL'S TRIANGLE - named after French mathematician Blaise Pascal

(1623-1662)

Row		SUM
1st	1	$1 = 2^0$
2nd	1 1	$2 = 2^1$
3rd	1 2 1	$4 = 2^2$
4th	1 3 3 1	$8 = 2^3$
5th	1 4 6 4 1	$16 = 2^4$
6th	1 5 10 10 5 1	$32 = 2^5$
7th	1 6 15 20 15 6 1	$64 = 2^6$
nth	$1 (n-1) \dots (n-1) 1$	2^{n-1}

Pascal's Δ is SYMMETRICAL.

In the $(n+1)^{\text{th}}$ row, the $(r+1)^{\text{th}}$ number is
 (the number of combinations of n objects
 taken r at a time)

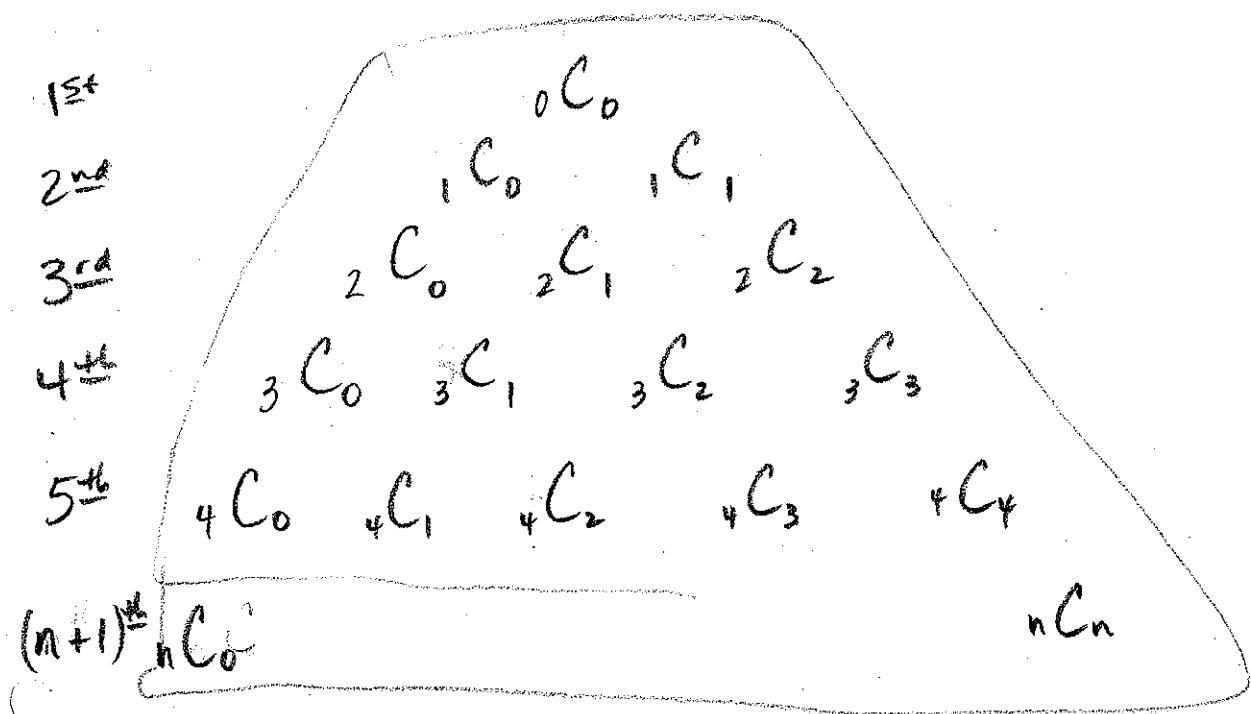
$$nC_r$$

~~eg:~~ 4th row, third number? $\rightarrow {}^3C_2 = \boxed{3}$

~~eg:~~ 5th row, 2nd number? $\rightarrow {}^4C_1 = \boxed{4}$

~~eg:~~ 7th row, 1st number? $\rightarrow {}^6C_0 = \boxed{1}$

Every number in Pascal's Δ can be written as
 a combination:



- Pascal's Δ can be used to expand binomial powers:

e.g. Expand the following:

a) $(x+y)^2 = \underline{x^2 + 2xy + y^2}$

b) $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

NOTE:

① The coefficients of the terms of the binomial $(x+y)^n$ are the numbers in the $(n+1)^{\text{th}}$ row of Pascal's Δ . ($nC_0, nC_1, nC_2, \dots, nC_n$)

② The powers of x descend from $x^n \rightarrow x^{1^{\text{st}} \text{ term of binomial } (x+y)}$

③ The powers of y ascend from $y^0 \rightarrow y^n$
 $y^0 \rightarrow$ 2nd term of binomial $(x+y)$

④ Row 1 of Δ represents $(x+y)^0$.

e.g.: Expand $(x+y)^5$

Using Δ : 6th row: $x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$

without Δ : $(x+y)^5 = {}_5C_0 x^5 y^0 + {}_5C_1 x^4 y^1 + {}_5C_2 x^3 y^2 + {}_5C_3 x^2 y^3 +$
 ${}_5C_4 x^1 y^4 + {}_5C_5 x^0 y^5$

$$= x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

Q2: Expand $(2x-3)^3$

$$= {}_3C_0 (2x)^3 (-3)^0 + {}_3C_1 (2x)^2 (-3)^1 + {}_3C_2 (2x)^1 (-3)^2 + {}_3C_3 (2x)^0 (-3)^3$$

$$= 1 \cdot 8x^3 \cdot 1 + 3(4x^2)(-3) + 3(2x)(9) + 1 \cdot 1(-27)$$

$$\boxed{= 8x^3 - 36x^2 + 54x - 27}$$

Binomial Theorem

$$(a+b)^n = {}_nC_0 a^n b^0 + {}_nC_1 a^{n-1} b^1 + {}_nC_2 a^{n-2} b^2 + \dots + {}_{n-1}C_{n-1} a^1 b^{n-1} + {}_nC_n a^0 b^n$$

General term: $\boxed{{}_nC_k a^{n-k} b^k}$

Specific Term of a Binomial Expansion

For $(a+b)^n$, the specific term is:

$$\boxed{t_{k+1} = {}_nC_k a^{n-k} b^k}$$

FYI: $t_1 = {}_nC_0 a^n b^0$
($k=0$)

Q3: Find the 6th term of $(x-2y)^{10}$

$t_6 \therefore k=5, n=10, a=x, b=-2y$

$$t_6 = ({}_{10}C_5)(x^5)(-2y)^5 \rightarrow = 252x^5(-32y^5)$$

$$= \frac{(10!)}{5!} x^5 (-32y^5) \rightarrow = -8064x^5y^5$$

Eg 4: Find the coefficient of x^2 in the expansion of $(\sqrt{x} - 2)^{10}$.

$$n = 10$$

$$a = \sqrt{x}$$

$$b = -2$$

$$k = ?$$

$$t_{k+1} = n C_k a^{n-k} b^k$$

$$t_{k+1} = ({}_{10} C_k) (\sqrt{x})^{10-k} (-2)^k$$

$$(\sqrt{x})^{10-k} = x^2 \quad (10-k) \log \sqrt{x} = 2 \log x$$

$$(x^{\frac{1}{2}})^{10-k} = x^2 \quad (10-k) \log (x^{\frac{1}{2}}) = 2 \log x$$

$$x^{5-\frac{k}{2}} = x^2 \quad 5 - \frac{k}{2} \log x = 2 \log x$$

$$5 - \frac{k}{2} = 2$$

$$5 - \frac{k}{2} = 2$$

$$3 = \frac{k}{2}$$

$$k = 6$$

$$t_{k+1} = t_7 = {}_{10} C_6 (\sqrt{x})^4 (-2)^6$$

$$= 210 x^2 (64)$$

$$= 13440 x^2$$

coeff. is 13440

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Q5: Find the constant term in the expansion of $(2x^2 + \frac{1}{x})^9$.

$$n=9 \quad a=2x^2 \quad b=\frac{1}{x} \quad k=?$$

$$t_{k+1} = n C_k \quad a^{n-k} \quad b^k \quad * \text{constant term has } x^0 \text{ as variable}$$

$$t_{k+1} = 9 C_k \quad (2x^2)^{9-k} \quad \left(\frac{1}{x}\right)^k$$

ignore coefficients, we'll deal with them later:

$$(x^2)^{9-k} (x^{-1})^k = x^0$$

$$(x^{18-2k})(x^{-k}) = x^0$$

$$18-2k + (-k) = 0$$

$$18-3k = 0$$

$$k=6$$

7th term is constant!

$$t_7 = 9 C_6 \quad (2x^2)^3 \quad \left(\frac{1}{x}\right)^6$$

$$= 84 (8x^6) \left(\frac{1}{x^6}\right)$$

$$\boxed{= 672}$$

p. 336 - 337
1-14

Ch. 7.5 - Pathway Problems

- used in the networking/communication field and relies upon the principles established in Permutations and Binomial Expansion.

Most Pathway Problems can be solved in two ways:

① Using the Permutations Principle:

$$\frac{n!}{n_1! n_2! \dots n_k!}$$

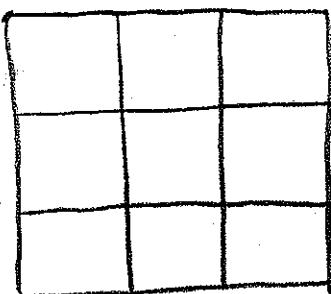
where $n_1 + n_2 + n_3 + \dots + n_k = n$

of repeats of a particular type

② Using the pattern observed in Pascal's Δ.

e.g. 1: How many different ways can one travel from point A to point B if they can only travel east and south?

A



Method ①:

Whichever path chosen requires 3 blocks E and 3 blocks S

EEESSS $n = 6$

B

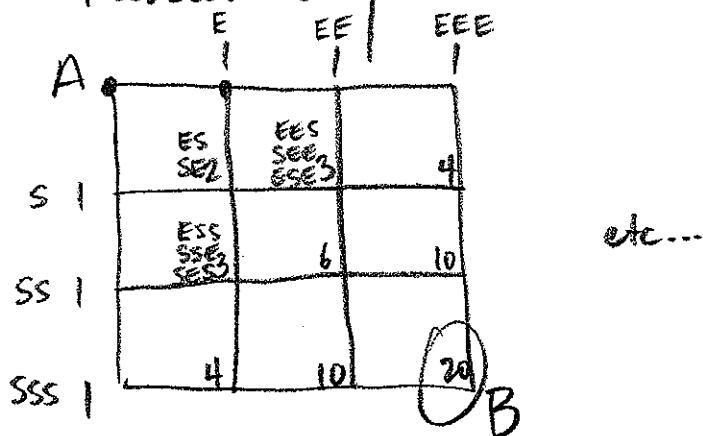
E repeats $\times 3$
S repeats $\times 3$

$$\frac{n!}{n_1! n_2!} = \frac{6!}{3! 3!}$$

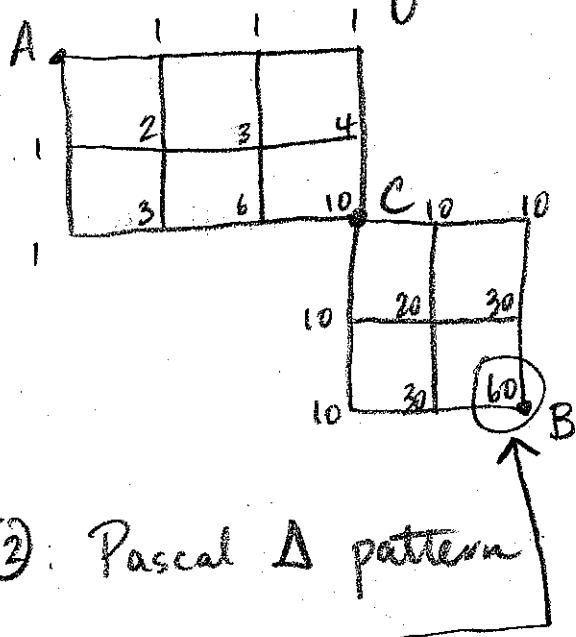
$$= 20$$

OR ...

Pascal Δ pattern:



q2: How many different routes can be taken to travel from A to B by only going down and to the right?



Method (2): Pascal Δ pattern

Method (1): involves two steps:

- label point C

- a) from A to C } Use Fund.
- b) from C to B } Counting Principle

A to C: 3 across, 2 down

$$= \frac{5!}{3!2!} = 10$$

C to B: 2 across, 2 down

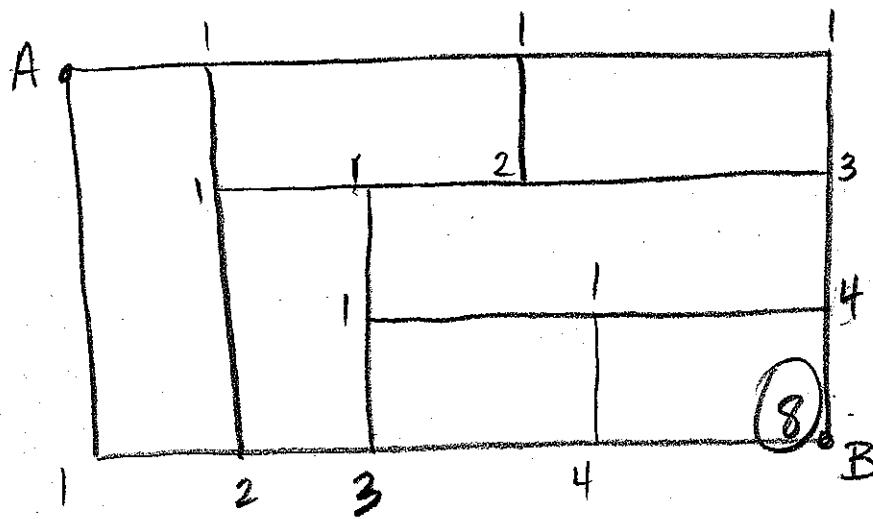
$$= \frac{4!}{2!2!} = 6$$

$$\therefore A \text{ to } B = \underbrace{10 \times 6}_{\text{F. Count. Principle}} = \boxed{60}$$

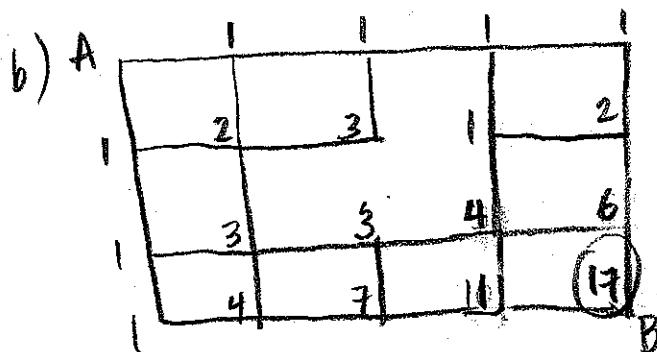
Note: Permutations will not work
(Method ①)

for rectangles with missing segments.

~~q3:~~ a) How many different paths are there from A to B if one can only move right and down?



Have to use
Pascal Δ
Pattern!



P. 339-340
#1, 2

Review:

1-50