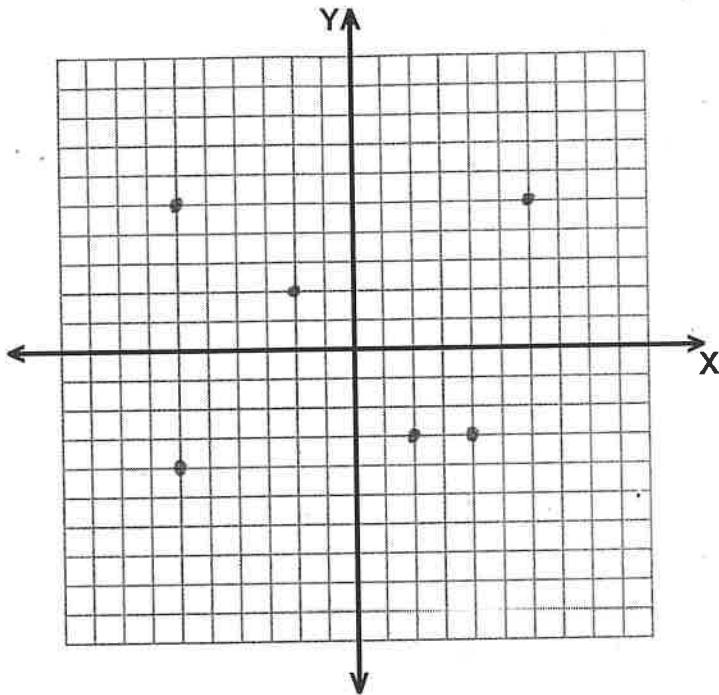


Ch. 2.1 - Functions and Relations

- functions and relations are best visualized by a GRAPH (coordinate (Cartesian) plane).

Relation - any set of ORDERED PAIRS (ie. a set of x/y points on a graph).



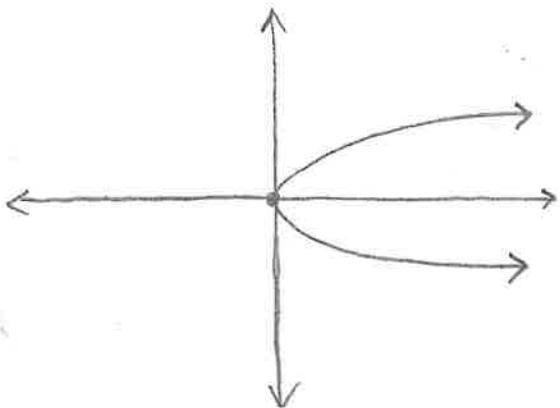
Domain - the set of x-values (Input) of all the points.

$$D: x = \underline{-6, -2, 2, 4, 6}$$

Range - the set of y-values (Output) of all the points.

$$R: y = \underline{-4, -3, 2, 5}$$

e.g. 1: Given the following relation, provide the DOMAIN and RANGE:



D: $x \geq 0$

R: $y \in \mathbb{R}$

"belongs to"

the set of REAL numbers

- without a graph, however, determining the domain and range is more challenging.
- a couple of hints, therefore:
 - i) the square root (or, even index root) of a negative number is undefined in \mathbb{R} .
 - ii) any numerator divided by zero is undefined.

q2: Determine the domain and range of the following functions:

$$a) y = -\sqrt{x-3} + 2 \quad b) y = \frac{2}{x+1}$$

D: $x \geq 3$

R: $y \leq 2$

* the lowest $\sqrt{x-3}$ will be is 0.

D: $x \neq -1$

R: $y \neq 0$

* 2 over "something" will always be "something"

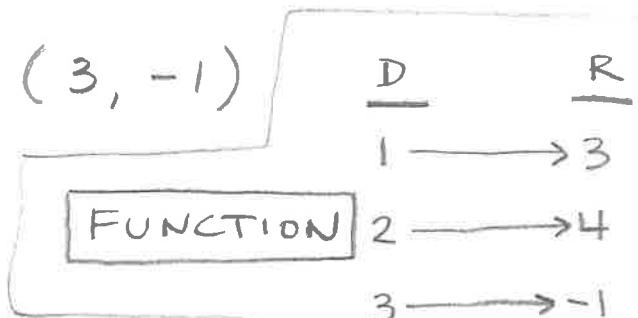
- Sometimes, domain/range may be referred to as RESTRICTIONS upon x/y respectively.

Function - a relation whereby, for every value of the domain (ie. x-value), there exists ONE and only ONE value for the range (ie. y-value).

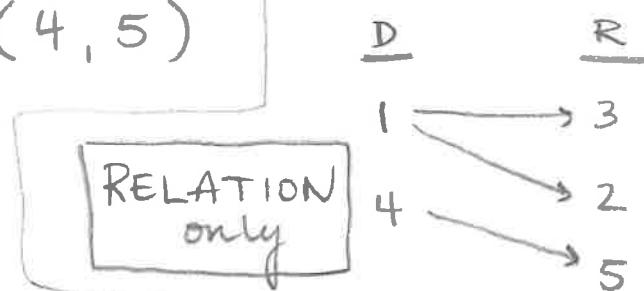
i.e. ANY x-value CAN ONLY 'HAVE' ONE y-value.

eg3: Which of the following relations are FUNCTIONS?

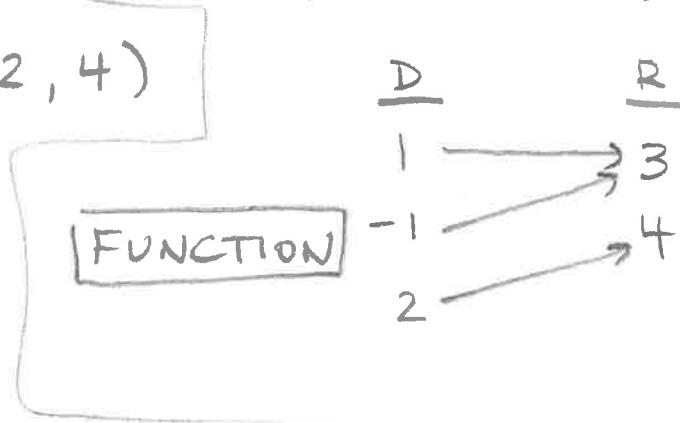
- a) $(1, 3)$, $(2, 4)$, $(3, -1)$



- b) $(1, 3)$, $(1, 2)$, $(4, 5)$



- c) $(1, 3)$, $(-1, 3)$, $(2, 4)$



$$d) \quad x^2 + y^2 = 9$$

$$y^2 = 9 - x^2$$

$$\sqrt{y^2} = \sqrt{9 - x^2}$$

$$y = \pm \sqrt{9 - x^2}$$

RELATION

one x -value
creates TWO
 y -values

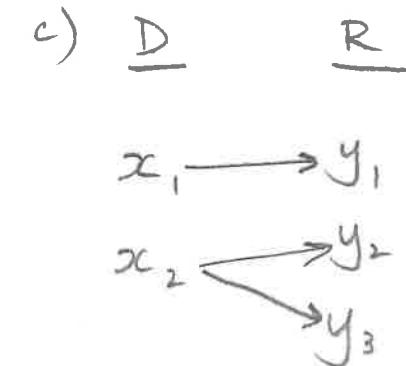
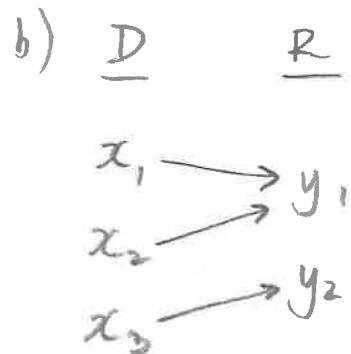
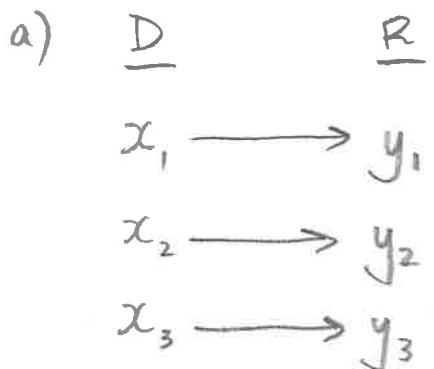
* if y is raised to an even power,
NOT a fxn.

One-to-One Function - a fxn in which
every one value of the domain (ie. x -value)
is associated with only one value of
the range (ie. y -value), AND vice versa.

ie. each x 'maps' to ONE distinct y .
AND

each y 'maps' to ONE distinct x .

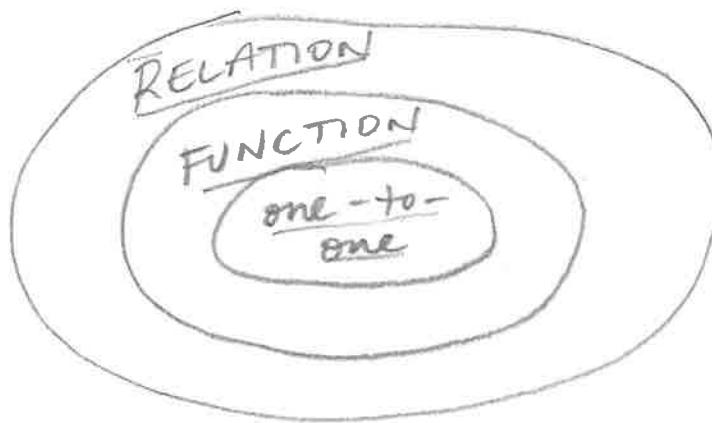
eg 4: Given the following 'maps', indicate whether the system represents a one-to-one fcn, a fcn, or a relation:



ONE-TO-ONE

FXN.

RELATION



Line Tests for Functions

- used in conjunction with GRAPHS.

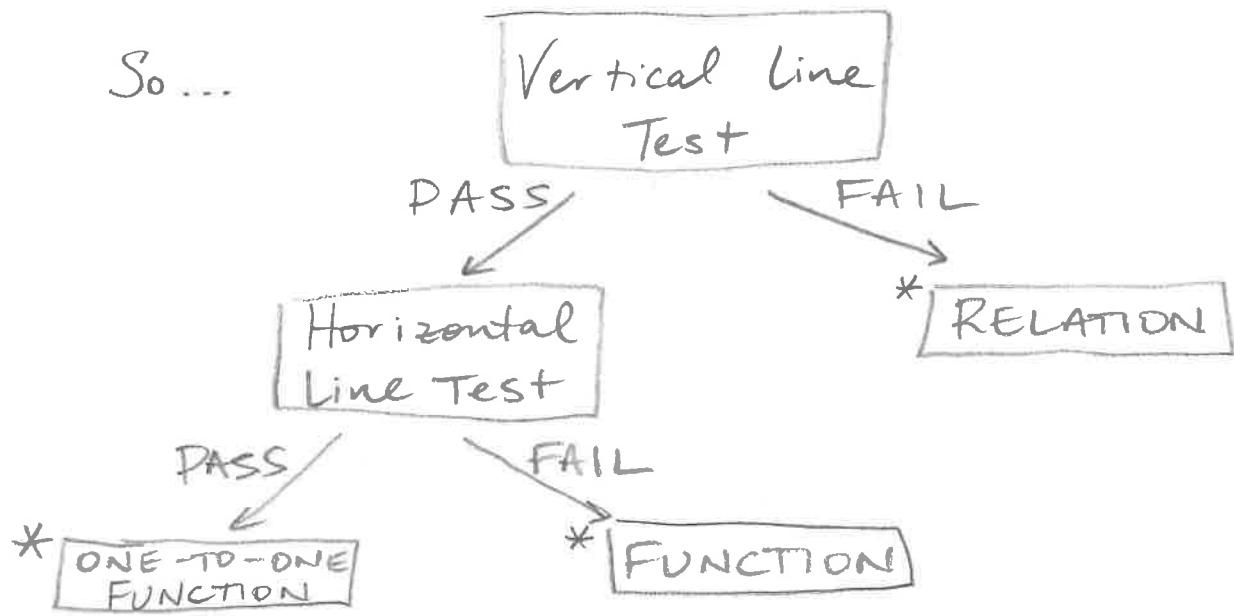
i) Vertical Line Test:

an equation defines y as a FUNCTION of x if and only if every vertical line drawn in the coordinate plane intersects the graph of the equation only ONCE

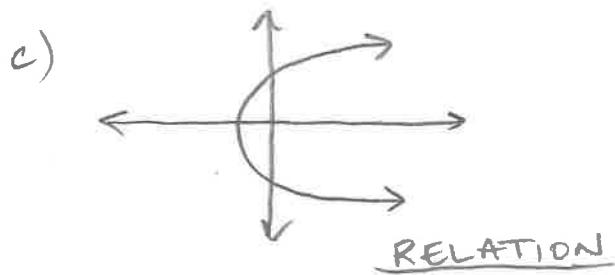
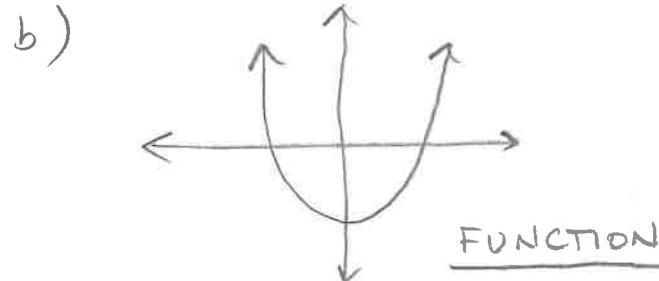
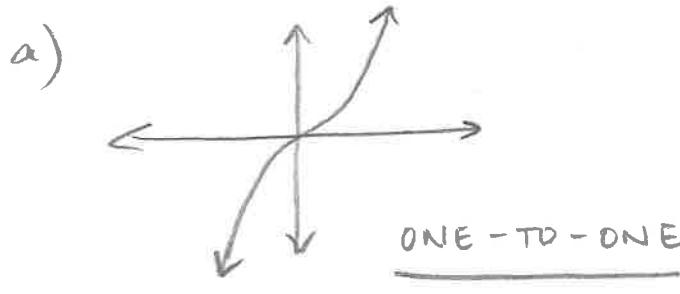
Horizontal Line Test:

a function, y , is a one-to-one function if and only if every horizontal line drawn in the coordinate plane intersects the function only ONCE.

So...



Q5: One-to-One Function, Function, or Relation?



Hwk: see eg. 6 on p. 52

* \circ = exclusive point
• = inclusive point

P 53-54 # 1-3

Typo: p.403 2; neither

Ch. 2.2 - Arithmetic Combinations of Functions

Let f and g each be functions. Then:

i) the SUM of f and g = $f + g$ OR,

$$\begin{aligned}\text{the SUM of } f \text{ and } g, \text{ as defined by } x, &= \underline{(f+g)(x)} \\ &= \underline{f(x) + g(x)};\end{aligned}$$

ii) the DIFFERENCE of f and g = $f - g$ OR,

$$\begin{aligned}\text{the DIFFERENCE of } f \text{ and } g, \text{ as defined by } x, &= \underline{(f-g)(x)} \\ &= \underline{f(x) - g(x)};\end{aligned}$$

iii) the PRODUCT of f and g = fg OR,

the PRODUCT of f and g , as defined by x ,

$$= \underline{(fg)(x)} = \underline{f(x) \cdot g(x)};$$

iv) the QUOTIENT of f and g = $\frac{f}{g}$ ($g \neq 0$) OR,

the QUOTIENT of f and g , as defined by x ,

$$= \underline{\left(\frac{f}{g}\right)(x)} = \underline{\frac{f(x)}{g(x)}} \quad \underline{g(x) \neq 0}.$$

x -values serve as the INPUT (DOMAIN) of the functions; $f(x)$ and $g(x)$ serve as the OUTPUT (RANGE).

q1: Let $f(x) = 3x + 6$ and $g(x) = x^2 - 4$,

Find:

a) $(f + g)(x) = f(x) + g(x) = (3x + 6) + (x^2 - 4)$

$$= x^2 + 3x + 2$$

OR

$$= (x+2)(x+1)$$

b) $(f - g)(x) = f(x) - g(x) = (3x + 6) - (x^2 - 4)$

$$= 3x + 6 - x^2 + 4$$
$$= -x^2 + 3x + 10$$
$$= -1(x^2 - 3x - 10)$$
$$= -1(x - 5)(x + 2)$$

c) $(fg)(x) = f(x) \cdot g(x) = (3x + 6)(x^2 - 4)$

$$= 3x^3 - 12x + 6x^2 - 24$$
$$= 3x^3 + 6x^2 - 12x - 24$$
$$= 3(x^3 + 2x^2 - 4x - 8)$$

d) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{3x + 6}{x^2 - 4} = \frac{3(x + 2)}{(x + 2)(x - 2)}$

$$= \frac{3}{x-2}$$

$x \neq \pm 2$

eg2: Compute each of the following, given:

$$f(x) = 2x + 1 \quad g(x) = x^2 - 2x + 1 \quad h(x) = x^3 \quad k(x) = 2$$

a) $(f + g)(1)$

$$\begin{aligned} &= f(1) + g(1) \\ &= (2(1) + 1) + ((1)^2 - 2(1) + 1) \\ &= (2 + 1) + (1 - 2 + 1) \\ &= 3 + 0 \\ &= \boxed{3} \end{aligned}$$

b) $(h - k)(-2)$

$$\begin{aligned} &= h(-2) - k(-2) \\ &= (-2)^3 - (2) \\ &= -8 - 2 \\ &= \boxed{-10} \end{aligned}$$

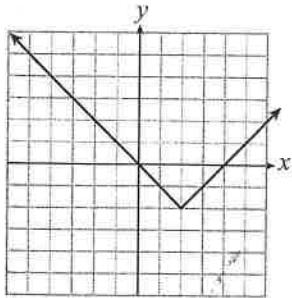
c) $\left(\frac{kg}{h}\right)(3)$

$$\begin{aligned} &= \frac{k(3) \cdot g(3)}{h(3)} = \frac{(2) \cdot ((3)^2 - 2(3) + 1)}{(3)^3} \\ &= \frac{2 \cdot (9 - 6 + 1)}{27} \\ &= \frac{2 \cdot 4}{27} \\ &= \boxed{\frac{8}{27}} \end{aligned}$$

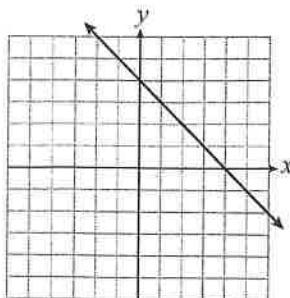
$$\begin{aligned}
 \text{d) } f \cdot k(0) - h \cdot g(-4) &= [f(0) \cdot k(0)] - [h(-4) \cdot g(-4)] \\
 &= ((2(0)+1) \cdot (2)) - ((-4)^3 \cdot ((-4)^2 - 2(-4) + 1)) \\
 &= (1 \cdot 2) - (-64) \cdot (16 + 8 + 1) \\
 &= 2 - (-64)(25) \\
 &= 2 - (-1600) \\
 &= \boxed{1602}
 \end{aligned}$$

$$\begin{aligned}
 \text{e) } (h \cdot (f+g))(2) &= h(2) \cdot (f(2) + g(2)) \\
 &= (2)^3 \cdot ((2(2)+1) + ((2)^2 - 2(2) + 1)) \\
 &= 8 \cdot ((4+1) + (4-4+1)) \\
 &= 8 \cdot (5+1) \\
 &= 8 \cdot 6 \\
 &= \boxed{48}
 \end{aligned}$$

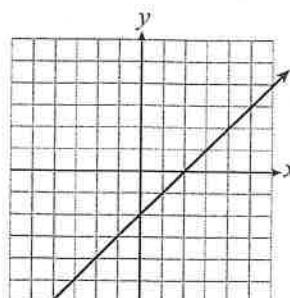
3: Use the graphs of f , g , and h to evaluate the following:



$$y = f(x)$$



$$y = g(x)$$



$$y = h(x)$$

a) $(f + g)(3) = f(3) + g(3) = (-1) + (1) = \boxed{0}$

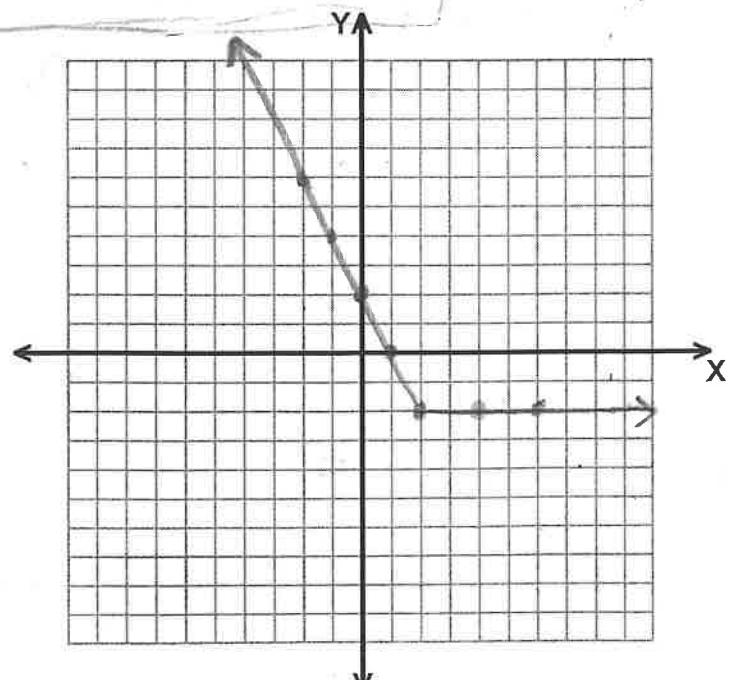
b) $\left(\frac{h}{g}\right)(4) = \frac{h(4)}{g(4)} = \frac{(2)}{(0)} = \boxed{\text{UNDEFINED}}$

c) $(fgh)(1) = f(1) \cdot g(1) \cdot h(1) = (-1)(3)(-1) = \boxed{3}$

d) Graph $(f - h)(x) = f(x) - h(x)$

$$x | f(x) - h(x)$$

0	$0 - (-2) = 2$ $(0, 2)$
-1	$1 - (-3) = 4$ $(-1, 4)$
-2	$2 - (-4) = 6$ $(-2, 6)$
1	$-1 - (-1) = 0$ $(1, 0)$
2	$-2 - (0) = -2$ $(2, -2)$
4	$0 - (2) = -2$ $(4, -2)$
6	$2 - (4) = -2$ $(6, -2)$



* recall PIECEWISE Theory :

When $x \leq 2$, $f(x) = -x$ ($h(x) = x - 2$)
always

So, when $x \leq 2$,

$$\begin{aligned}f(x) - h(x) &= (-x) - (x - 2) \\&= -2x + 2\end{aligned}$$

When $x \geq 2$, $f(x) = x - 4$

So, when $x \geq 2$,

$$\begin{aligned}f(x) - h(x) &= (x - 4) - (x - 2) \\&= -4 + 2 \\&= -2\end{aligned}$$

p. 57 # 1-8

typo: p. 403 # 3g $(4x^2 - 2x - 2)$

typo: p. 403 # 5c
p. 404 # 6e } graph correction

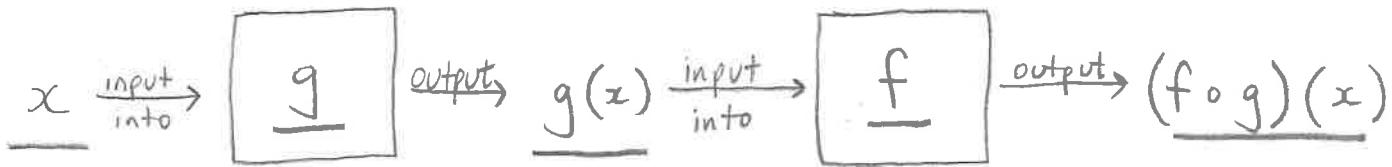
Ch. 2.3 - Composite Functions

- relies upon the concept of algebraic
SUBSTITUTION.

Let f and g be functions defined by x .

Then f composed with g

$$\begin{aligned} &= f \circ g \\ &= (f \circ g)(x) \\ &= f(g(x)) \\ &\text{ie. "f of g of x"} \end{aligned}$$



Thus, x must be in g 's domain and $g(x)$ (g 's range) must be in f 's domain for $(f \circ g)(x)$ to be DEFINED.

FORMAL DEFINITION:

The composite function $f \circ g$ of two functions f and g (each defined by x) is defined by:

$(f \circ g)(x) = f(g(x))$ for all x in the domain of g such that $g(x)$ is in the domain of f .

eg1: Given $f = \{(1, d), (3, e)\}$ and

$$g = \{(a, 1), (b, 3), (c, 5)\},$$

find $(f \circ g)(x)$.

$$(f \circ g)(x) = f(g(x))$$

There are only 3 defined 'inputs' for g : a , b , and c .

$$f(g(a)) = f(1) = d$$

$$f(g(b)) = f(3) = e$$

$$f(g(c)) = f(5) = \emptyset \text{ (undefined)}$$

$$\therefore (f \circ g)(x) = \{(a, d), (b, e)\}$$

eg2. If $f(x) = 1 - x^2$ and $g(x) = 2x + 3$, find:

a) $(f \circ g)(x)$

$$= f(g(x))$$

$$= f(2x + 3)$$

$$= (1 - (2x+3)^2)$$

$$= 1 - (4x^2 + 12x + 9)$$

$$= -4x^2 - 12x - 8$$

b) $(g \circ f)(x)$

$$= g(f(x))$$

$$= g(1 - x^2)$$

$$= 2(1 - x^2) + 3$$

$$= 2 - 2x^2 + 3$$

$$= -2x^2 + 5$$

not the same!?

OR

$$= -4(x^2 + 3x + 2)$$

Eg3: Given $f(x) = x^2$ and $g(x) = 2x - 1$,
find:

$$(f \circ g)(-2)$$

$$\begin{aligned} &= f(g(-2)) = f(2(-2) - 1) \\ &= f(-5) \\ &= (-5)^2 \\ &= 25 \end{aligned}$$

Eg4: Given $f(x) = x^2 - 1$ and $g(x) = \sqrt{x}$,
find:

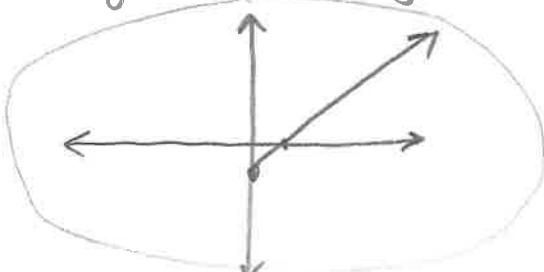
a) $(f \circ g)(x)$ and state any restrictions on
its domain.

$$\begin{aligned} &= f(g(x)) \\ &= f(\sqrt{x}) \quad x \geq 0 \\ &= (\sqrt{x})^2 - 1 \end{aligned}$$

$$= x - 1 \quad \text{no 'new' restrictions}$$

so, $x \geq 0$

b) Sketch a graph of $y = (f \circ g)(x)$



Q5: Given $f(x) = \frac{x}{x-1}$ and $g(x) = \frac{1}{x+1}$, find

a) $(f \circ g)(x)$ and its domain

b) $(g \circ f)(x)$ and its domain

$$= f(g(x))$$

$$= f\left(\frac{1}{x+1}\right)$$

$$= \frac{\left(\frac{1}{x+1}\right)}{\left(\frac{1}{x+1}\right)-1} \quad x \neq -1$$

$$= \frac{\left(\frac{1}{x+1}\right)}{\left(\frac{1}{x+1}\right)-\left(\frac{x+1}{x+1}\right)}$$

$$= \frac{\left(\frac{1}{x+1}\right)}{\left(\frac{-x}{x+1}\right)}$$

$$= \boxed{-\frac{1}{x}} \quad x \neq 0$$

$$D: x \neq -1, 0$$

$$= g(f(x))$$

$$= g\left(\frac{x}{x-1}\right)$$

$$= \frac{1}{\left(\frac{x}{x-1}\right)+1} \quad x \neq 1$$

$$= \frac{1}{\left(\frac{x}{x-1}\right)+\left(\frac{x-1}{x-1}\right)}$$

$$= \frac{1}{\left(\frac{2x-1}{x-1}\right)}$$

$$\boxed{= \frac{x-1}{2x-1}} \quad x \neq \frac{1}{2}$$

$$\boxed{D: x \neq \frac{1}{2}, 1}$$

eg 6: Use the graph of $f(x)$ and $g(x)$ to find:

a) $(f \circ g)(1)$

$$= f(g(1))$$

$$= f(-3)$$

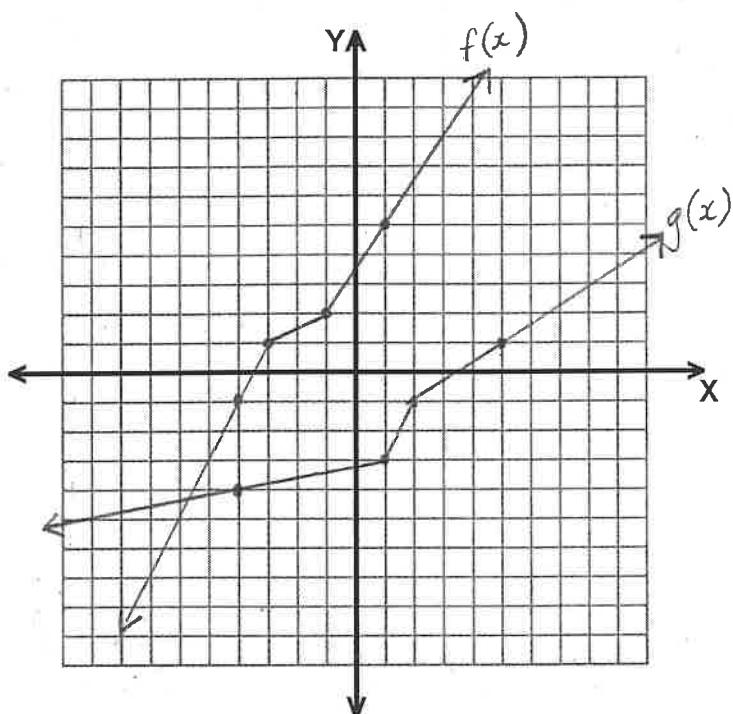
$$= \boxed{1}$$

b) $(g \circ g)(5)$

$$= g(g(5))$$

$$= g(1)$$

$$= \boxed{-3}$$



Decomposing a Composite Function

- Ask yourself:

- what expression of the composition is the 1^{st} input value, and what expression is the 2^{nd} input value?

eg 7: Given $h(x) = \sqrt{x-2}$, find two functions f and g such that $(f \circ g)(x) = h(x)$.

$$(f \circ g)(x) = f(g(x)) \quad g(x) \text{ is the } 1^{\text{st}} \text{ input}$$

$$\boxed{g(x) = x - 2}$$

$$f(x) \text{ is the } 2^{\text{nd}} \text{ input}$$

$$\boxed{f(x) = \sqrt{x}}$$

q8: Given the following expressions for $h(x)$,
find functions f and g such that
 $(f \circ g)(x) = h(x)$

a) $h(x) = \sqrt[3]{x+5}$

$$(f \circ g)(x) = f(g(x))$$

$g(x) = x + 5$ $f(x) = \sqrt[3]{x}$

b) $h(x) = (\sqrt{x} + 1)^3 - 2$

$$(f \circ g)(x) = f(g(x))$$

$g(x) = \sqrt{x} + 1$ $f(x) = x^3 - 2$

* Note: this is just one strategy;
SOLUTIONS may vary

p. 66 # 1-9, 11-12

* ENRICHMENT next page!

Enrichment for future Calculus students:

Compute $\frac{f(x+h) - f(x)}{h}$ ($h \neq 0$) when
 $f(x) = 2x^2 + 3$

$$= \frac{2(x+h)^2 + 3 - (2x^2 + 3)}{h}$$

$$= \frac{2(x^2 + 2xh + h^2) + 3 - 2x^2 - 3}{h}$$

$$= \frac{2x^2 + 4xh + 2h^2 + 3 - 2x^2 - 3}{h}$$

$$= \frac{4xh + 2h^2}{h}$$

$$= \frac{h(4x + 2h)}{h}$$

$$= \boxed{4x + 2h}$$

as $h \rightarrow 0$, $4x + 2h = 4x$
(CALCULUS theory)

Try #10 in homework set.
(p. 70)

Ch. 2.4 - Transformations of Graphs

- the graphs of functions are able to undergo TRANSFORMATIONS.
- these transformations have the ability to alter a function's:
 - position;
 - end behaviour;
 - shape.

Types of Transformations:

1. TRANSLATIONS - vertical and horizontal shifts of a graph.
 - alters position.
2. REFLECTIONS - vertical (over x-axis) and horizontal (over y-axis) 'foldings' of the graph.
 - alters end behaviour
3. EXPANSIONS / COMPRESSIONS -
 - vertical and horizontal distortions of the graph.
 - alters shape.

For a BASIC function, $y = f(x)$, these transformations are governed by the a , b , c , and d values as seen in:

$$\underline{y = a f(b(x-c)) + d}$$

Note: a function is deemed BASIC when

$$a = \underline{1}, b = \underline{1}, c = \underline{0}, d = \underline{0}$$

$$\underline{\underline{y = 1 f(1(x-0)) + 0}}$$

\downarrow

$$\underline{\underline{y = f(x)}}$$

Also note: - b and c affect the DOMAIN.

- a and d affect the RANGE.

Translations

- a translation shifts a graph in the x and/or y direction without altering the graph's end behaviour or shape.
- Two types: Vertical and Horizontal

a) VERTICAL TRANSLATIONS - rely upon d .

i) if $d > 0$, then $y = f(x) + d$ is $y = f(x)$ shifted UP d units.

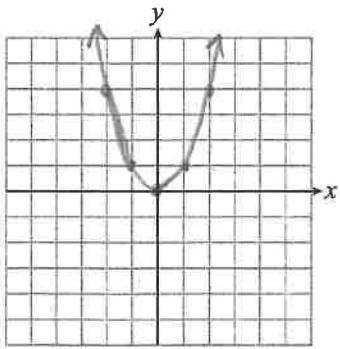
ie. $y = f(x) + d$

ii) if $d < 0$, then $y = f(x) + d$ is $y = f(x)$ shifted DOWN d units.

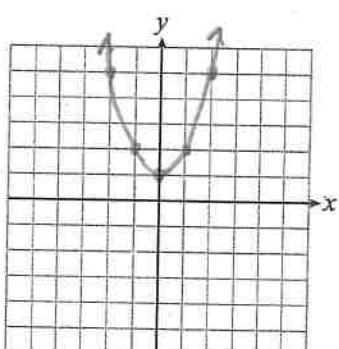
ie. $y = f(x) - d$

iii) if $d = 0$, then $y = f(x) + d \equiv y = f(x)$. (ie. no vertical shift)

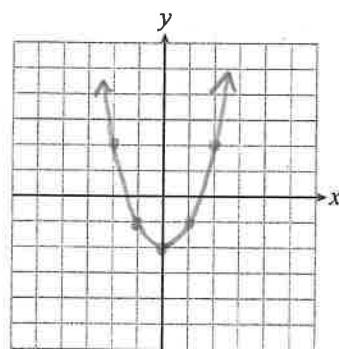
e.g. 1: Given $y = f(x) = x^2$, graph:



$$\begin{aligned}y &= f(x) \\y &= x^2\end{aligned}$$



$$\begin{aligned}y &= f(x) + 1 \\y &= x^2 + 1 * y &= f(x) \text{ shifted 1 UP}\end{aligned}$$



$$\begin{aligned}y &= f(x) - 2 \\y &= x^2 - 2 * y &= f(x) \text{ shifted 2 DOWN}\end{aligned}$$

b) HORIZONTAL TRANSLATIONS - rely upon c.

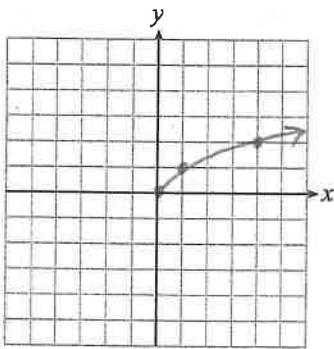
i) if $c > 0$, then $y = f(x-c)$ is $y = f(x)$ shifted RIGHT c units.
ie. $y = f(x-c)$

ii) if $c < 0$, then $y = f(x-c)$ is $y = f(x)$ shifted LEFT c units.

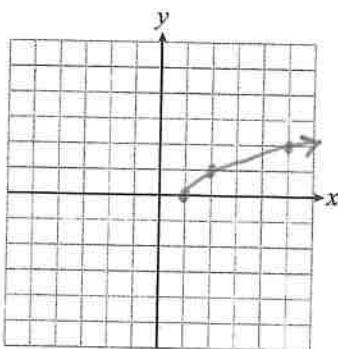
ie. $y = f(x+c)$

iii) if $c = 0$, then $y = f(x-c)$ is $y = f(x)$. (ie. no horizontal shift.)

eg2: Given $y = f(x) = \sqrt{x}$, graph:

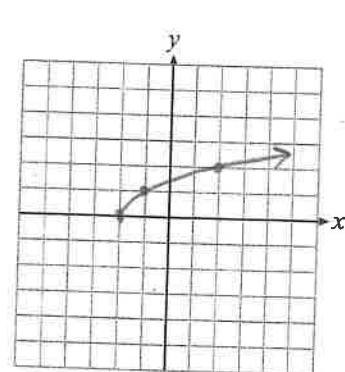


$$y = f(x)$$
$$y = \sqrt{x}$$



$$y = f(x-1)$$
$$y = \sqrt{x-1}$$

* $y = f(x)$ shifted 1 \textcircled{R}



$$y = f(x+2)$$
$$y = \sqrt{x+2}$$

* $y = f(x)$ shifted 2 \textcircled{L}

eg3: What transformation(s) must have occurred for $y = f(x)$ to become $y = f(x+2) + 4$?

2 units \textcircled{L}

4 units UP.

eg4: Write the equation of the function that is created when $y = f(x) = \sqrt{x}$ is shifted 4 units \textcircled{R} and 3 units DOWN.

$$y = f(x) \rightarrow y = f(x-4) - 3$$

$$y = \sqrt{x} \rightarrow \boxed{y = \sqrt{x-4} - 3}$$

eg 5: Given $y = f(x)$ possesses point $(-4, 3)$,
what point would exist on:

a) $y = f(x) + 4$?

$$(-4, 3) \rightarrow (-4, 3+4) \rightarrow \boxed{(-4, 7)}$$

b) $y = f(x+2)$?

$$(-4, 3) \rightarrow (-4+2, 3) \rightarrow \boxed{(-6, 3)}$$

c) $y = f(x-3) + 7$?

$$(-4, 3) \rightarrow (-4+3, 3+7) \rightarrow \boxed{(-1, 10)}$$

d) $y = f(x+1) - 2$?

$$(-4, 3) \rightarrow (-4+1, 3-2) \rightarrow \boxed{(-5, 1)}$$

p. 79 # 1 ace, 2 abc,
9 de, 10 ade.

Reflections

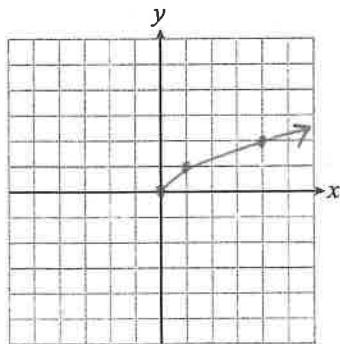
- a reflection has the ability to alter a graph's end behaviour.
 - reflections are always performed BEFORE translations.
 - Two types :
 - Vertical Reflections
 - over x-axis
 - Horizontal Reflections
 - over y-axis
- a) Vertical Reflections - rely upon 'a'
- more specifically, if 'a' is negative, there will be a reflection of the function over the x-axis.
- eg: Given $y = f(x)$, $y = -f(x)$ is a reflection of $y = f(x)$ over the x-axis.

b) Horizontal Reflections - rely upon b
- more specifically,
if b is negative,
there will be a
reflection of the
function over the
y-axis.

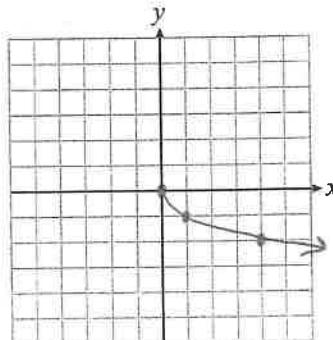
eg: Given $y = f(x)$, $y = f(-x)$ is a
reflection of $y = f(x)$ over the
y-axis.

* Note: $y = -f(-x)$ would be a reflection
of $y = f(x)$ over BOTH axes.

eg6: Given $y = f(x) = \sqrt{x}$, graph:



a)

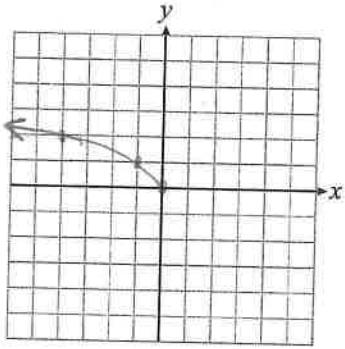


$$y = f(x)$$
$$y = \sqrt{x}$$

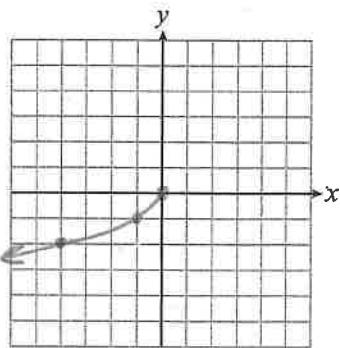
$$y = -f(x)$$
$$y = -\sqrt{x}$$

* a reflection of
 $y = f(x)$ over the
x-axis

b)



c)



$$y = f(-x)$$

$$y = \sqrt{-x}$$

* a reflection of
 $y = f(x)$ over the
 y -axis.

$$y = -f(-x)$$

$$y = -\sqrt{-x}$$

* a reflection of $y = f(x)$
over BOTH axes.

eg 7: Given $y = f(x)$ possesses $(3, -2)$, which point MUST be on :

a) $y = -f(x)$?

$$(3, -2) \rightarrow (3, (-2)(-1)) = (3, 2)$$

b) $y = f(-x)$?

$$(3, -2) \rightarrow ((3)(-1), -2) = (-3, -2)$$

c) $y = -f(-x)$?

$$(3, -2) \rightarrow ((3)(-1), (-2)(-1)) = (-3, 2)$$

eg 8: Write an equation for the function
 $y = x^2 - x$ if it is reflected over:

- a) the x -axis
- b) the y -axis
- c) BOTH axes

* first, set $y = x^2 - x = f(x)$

a) $y = f(x) \rightarrow y = -f(x)$

$$y = x^2 - x \rightarrow y = -(x^2 - x)$$

$y = -x^2 + x$

b) $y = f(x) \rightarrow y = f(-x)$

$$y = x^2 - x \rightarrow y = (-x)^2 - (-x)$$

$y = x^2 + x$

c) $y = f(x) \rightarrow y = -f(-x)$

$$y = x^2 - x \rightarrow y = -((-x)^2 - (-x))$$
$$y = -x^2 - x$$

$y = -x^2 - x$

Eg9: If $(-5, 3)$ exists on $y = f(x)$, then what point would exist on:

a) $y = -f(x-2) + 1$?

$$(-5, 3) \rightarrow (-5+2, (3)(-1) + 1)$$
$$= \boxed{(-10, -2)}$$

b) $y = f(-x) - 3$?

$$(-5, 3) \rightarrow ((-5)(-1), 3-3) = \boxed{(5, 0)}$$

c) $y = -f(-(x-1)) + 1$?

$$(-5, 3) \rightarrow ((-5)(-1) + 1, (3)(-1) + 1)$$
$$= \boxed{(6, -2)}$$

d) $y = f(-x+3) - 2$?

$$y = f(-(x-3)) - 2$$

$$(-5, 3) \rightarrow ((-5)(-1) + 3, 3-2)$$

$$= \boxed{(8, 1)}$$

p. 79

1 bdfgh, 2d-h, 9 a-c, f,
10 bcf.

Expansions / Compressions

- these types of transformations change the shape of (ie. distort) a graph.
- expansions / compressions are always performed BEFORE translations.
 - the order of applying them is interchangeable with reflections.

Two types : - Vertical Expansions / Compressions

- Horizontal Expansions / Compressions

a) Vertical Expansions / Compressions -

- rely upon ' a '.

Scenarios :

i) If $|a| > 1$, then $y = f(x)$ undergoes a vertical expansion by a factor of $|a|$.

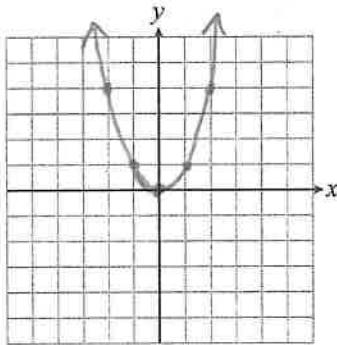
ii) If $0 < |a| < 1$, then $y = f(x)$ undergoes a vertical compression by a factor of $|a|$.

Eg: Given $y = f(x)$,

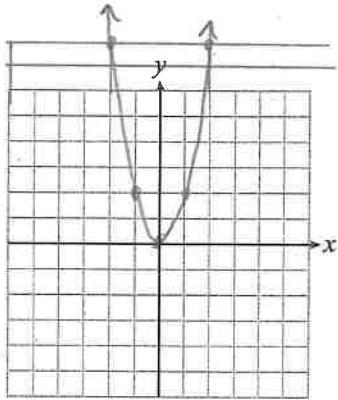
→ $y = 2f(x)$ is a vertical expansion of $y = f(x)$ by a factor of 2.

→ $y = \frac{1}{3}f(x)$ is a vertical compression of $y = f(x)$ by a factor of $\frac{1}{3}$.

eg 10: Given $y = f(x) = x^2$, graph:



a)

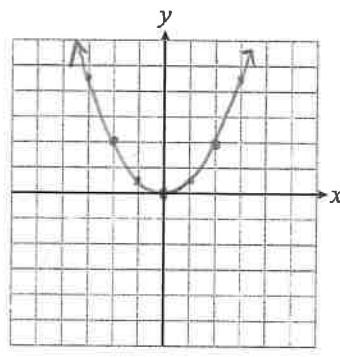


$$y = 2f(x)$$

$$\underline{y = 2x^2}$$

* $y = f(x)$ vertically expanded
by a factor of 2

b)



$$y = \frac{1}{2}f(x)$$

$$\underline{y = \frac{1}{2}x^2}$$

* $y = f(x)$ vertically compressed
by a factor of $\frac{1}{2}$.

b) Horizontal Expansions / Compressions -
- rely upon b.

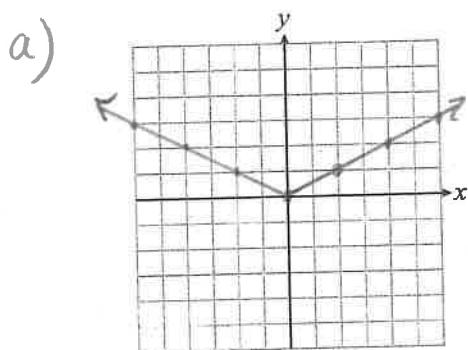
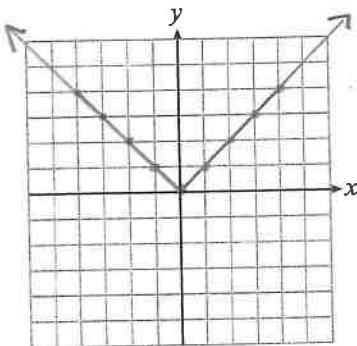
Scenarios:

- i) if $0 < |b| < 1$, then $y = f(x)$ undergoes a horizontal expansion by a factor of $|b|$.
- ii) if $|b| > 1$, then $y = f(x)$ undergoes a horizontal compression by a factor of $|b|$.

i.e. when $|b| > 1 \rightarrow$ horizontal expansion

when $0 < |b| < 1 \rightarrow$ horizontal compression

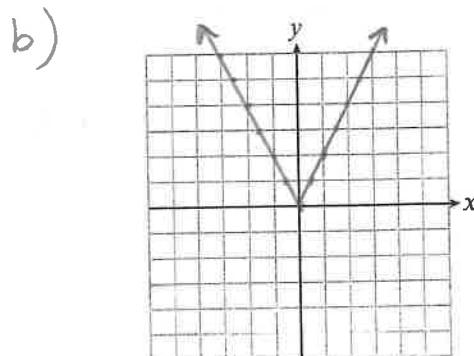
eg 11: Given $y = f(x) = |x|$, graph:



$$y = f\left(\frac{1}{2}x\right)$$

$$\underline{y = \left|\frac{1}{2}x\right|}$$

* $y = f(x)$ horizontally expanded by a factor of 2.



$$y = f(2x)$$

$$\underline{y = |2x|}$$

* $y = f(x)$ horizontally compressed by a factor of $\frac{1}{2}$.

eg 12: Describe the transformations that occur to change $y = f(x)$ into:

a) $y = 3f\left(\frac{1}{4}x\right)$

* vertical expansion factor 3.

* horizontal expansion factor 4.

b) $y = \frac{1}{5}f(2x)$

* vertical compression factor $\frac{1}{5}$.

* horizontal compression factor $\frac{1}{2}$.

eg 13: Given $y = \sqrt{x}$, write an equation representing a:

a) Vertical Expansion factor 2;

b) Horizontal Compression factor $\frac{1}{4}$;

c) Vertical Compression factor $\frac{1}{3}$;

d) Horizontal Compression factor 9.

Let $y = f(x) = \sqrt{x}$

a) $y = 2f(x) = 2\sqrt{x}$

c) $y = \frac{1}{3}f(x) = \frac{1}{3}\sqrt{x}$

b) $y = f(4x) = \sqrt{4x}$

d) $y = f\left(\frac{1}{9}x\right) = \sqrt{\frac{1}{9}x}$

y14: If $(5, -2)$ is a point on $y = f(x)$,
then which point must be on:

a) $y = 2f(4x)$?

$$(5, -2) \rightarrow ((5)(\frac{1}{4}), (-2)(2)) = \boxed{(\frac{5}{4}, -4)}$$

b) $y = \frac{1}{4}f(2x)$?

$$(5, -2) \rightarrow ((5)(\frac{1}{2}), (-2)(\frac{1}{4})) = \boxed{(\frac{5}{2}, -\frac{1}{2})}$$

c) $y = 3f(\frac{1}{2}x)$?

$$(5, -2) \rightarrow ((5)(2), (-2)(3)) = \boxed{(10, -6)}$$

d) $y = \frac{1}{2}f(\frac{x}{5})$?

$$y = \frac{1}{2}f(\frac{1}{5}x)$$

$$(5, -2) \rightarrow ((5)(5), -2(\frac{1}{2})) = \boxed{(25, -1)}$$

e) $y = |f(-x)|$?

$$(5, -2) \rightarrow ((5)(-1), -2) = (-5, -2) \rightarrow \boxed{(-5, 2)}$$

INVARIANT POINT - a point on a graph that remains unchanged (un-transformed) after a transformation is applied.

Notes: i) Translations result in NO invariant points;

ii) Reflections:

- any point that lies upon the LINE of REFLECTION is invariant.

iii) Expansions / Compressions:

a) Vertical Exp./Compr.:

- x-intercepts are invariant.

b) Horizontal Exp./Compr.:

- y-intercepts are invariant.

Sometimes, a vertical expansion and a horiz. compression are equivalent:

* depends on the specific function!

eg: If $y = f(x) = x^2$, then $y = 4f(x)$ is the same as $y = f(2x)$

$$y = 4x^2$$

SAME!

$$\begin{aligned}y &= (2x)^2 \\y &= 4x^2\end{aligned}$$

HOMEWORK: p. 79 →
the REST of it.

p. 47-48 handout

TYPo p. 407
#11 → $f(x) \geq 0$

Ch. 2.5 - Inverse Functions

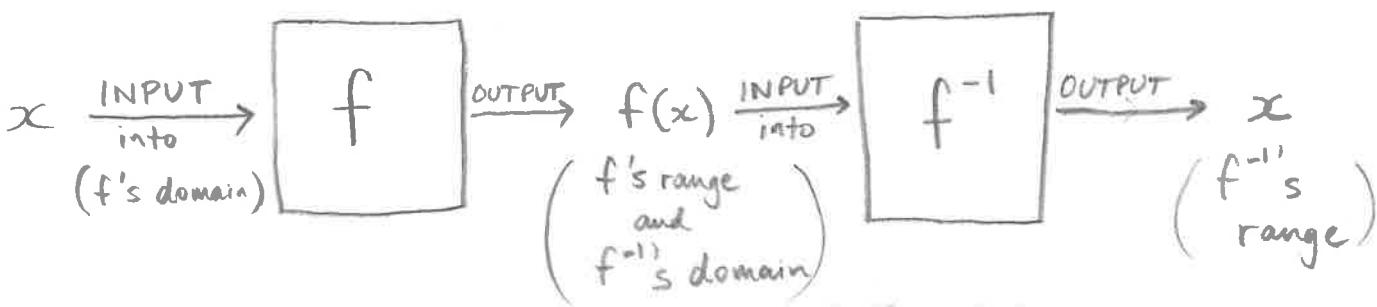
Two functions, f and g , are INVERSES of each other if and only if:

- i) $(f \circ g)(x) = f(g(x)) = \underline{x}$ AND
- ii) $(g \circ f)(x) = g(f(x)) = \underline{x}.$

* x MUST exist within both f and g 's domains.

→ informally, two functions are inverses of each other if each 'undoes' what the other 'does'.

Notation: $\underline{f^{-1}}$ means " f 's inverse."



* can move from ② to ① by altering language.

So... f 's domain 'matches' f^{-1} 's range, and f 's range 'matches' f^{-1} 's domain.

Also, two functions, f and g , are inverses of each other if :

- When the graph of $y=f(x)$ is reflected in the line $y=x$, it maps exactly onto $y=g(x)$, and vice versa.
- any point (a, b) on f exists as the point (b, a) on g .

Also, note :

- if f is a one-to-one function, then f^{-1} is a function.
- if f is not a one-to-one function, then f^{-1} is not a function.

Steps to determine an Inverse Function Algebraically:

1. Replace $f(x)$ with y (if necessary);
2. Interchange x and y (ie. change all x s to y s and all y s to x s);
3. Solve 'new' equation for y ;
4. Replace this 'new' y with $f^{-1}(x)$.

e.g. 1: Given $f(x) = 2x - 1$,

- Find $f^{-1}(x)$ algebraically.
- Graph $f(x)$ and $f^{-1}(x)$.
- Verify that they are, in fact, inverses of each other (in three different ways).

a) $f(x) = 2x - 1$

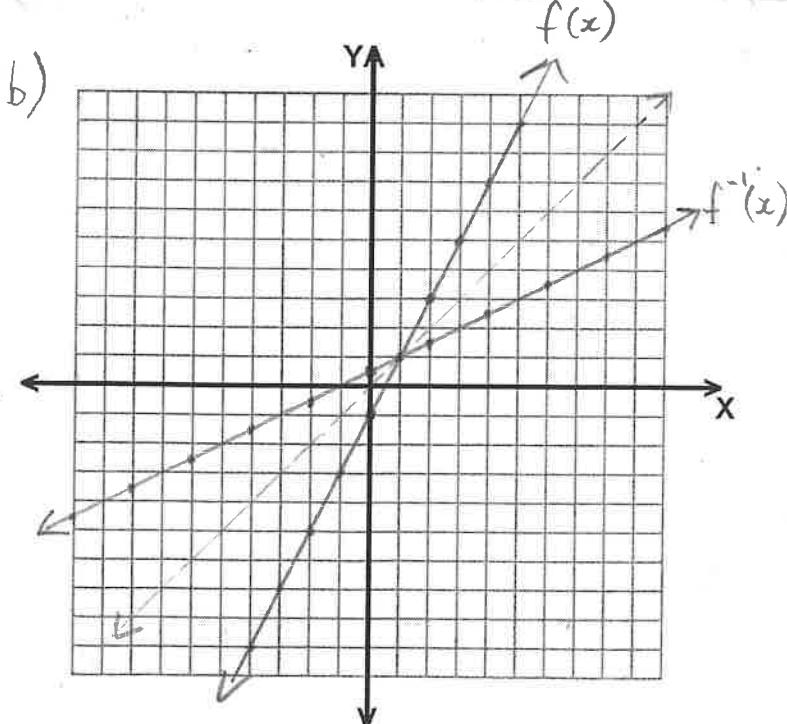
$$f: y = 2x - 1$$

$$f^{-1}: x = 2y - 1$$

$$2y = x + 1$$

$$y = \frac{x+1}{2} = \frac{1}{2}x + \frac{1}{2}$$

$$f^{-1}(x) = \frac{x+1}{2} = \frac{1}{2}x + \frac{1}{2}$$



*Note: (1, 1) is
an INVARIANT
point.

- c) i) Reflect $f(x)$ in $y=x$ and observe that it 'maps' on to $f^{-1}(x)$; vice versa is true as well.
- ii) 'Flip-flop' $f(x)$'s (x, y) coordinates and observe that they correspond to $f^{-1}(x)$'s (x, y) coordinates; vice versa is true as well.

iii) Prove $(f \circ f^{-1})(x) = x$ AND $(f^{-1} \circ f)(x) = x$

$$\begin{aligned}
 &= f(f^{-1}(x)) &&= f^{-1}(f(x)) \\
 &= f\left(\frac{x+1}{2}\right) &&= f^{-1}(2x-1) \\
 &= 2\left(\frac{x+1}{2}\right) - 1 &&= \frac{(2x-1) + 1}{2} \\
 &= x + 1 - 1 &&= \frac{2x}{2} \\
 &= \underline{\underline{x}} &&= \underline{\underline{x}}
 \end{aligned}$$

Verified!

Why is $(1, 1)$ invariant?

Because $(1, 1)$ 'flip-flopped' is still $(1, 1)$
OR

$(1, 1)$ lies on the line of reflection $y=x$.

eg2: Given $h(x) = \frac{x}{2x-3}$, find $h^{-1}(x)$.

Also, provide the domain and range for each of h and h^{-1} .

$$h: h(x) = \frac{x}{2x-3}$$

$$D: x \neq \frac{3}{2}$$

$$y = \frac{x}{2x-3}$$

$$R: y \neq \frac{1}{2}$$

$$h^{-1}: x = \frac{y}{2y-3}$$

$$x(2y-3) = y$$

$$2xy - 3x = y$$

$$3x = 2xy - y$$

$$3x = y(2x-1)$$

$$y = \frac{3x}{2x-1}$$

$$h^{-1}(x) = \frac{3x}{2x-1}$$

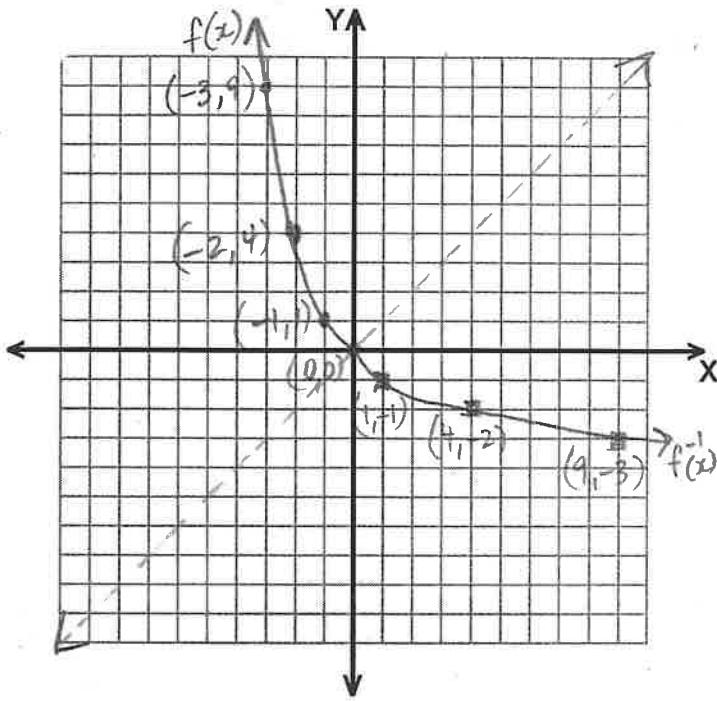
$$D: x \neq \frac{1}{2}$$

$$R: y \neq \frac{3}{2}$$

range of
'original'
same as
domain of
inverse

domain of
'original'
Same as
range of
inverse

eg3: Graph $f(x) = x^2$, where $x \leq 0$. Then, graph the inverse of $f(x)$.



* notice:

(0,0) is INVARIANT

eg4: Given $g(x) = \sqrt{x-1}$, find $g^{-1}(x)$. Find the domain and range for both g and g^{-1} .

$$g: g(x) = \sqrt{x-1} \quad D: x \geq 1$$

$$y = \sqrt{x-1} \quad R: y \geq 0$$

$$g^{-1}: x = \sqrt{y-1}$$

$$(\sqrt{y-1})^2 = x^2$$

$$y-1 = x^2$$

$$y = x^2 + 1$$

$$\boxed{g^{-1}(x) = x^2 + 1}$$

$$D: x \geq 0$$

$$R: y \geq 1$$

Algebraically:

$$f: f(x) = x^2 \quad D: x \leq 0$$

$$y = x^2 \quad R: y \geq 0$$

$$f^{-1}: x = y^2$$

$$\sqrt{y^2} = \sqrt{x}$$

$$y = \pm \sqrt{x} \quad R: y \leq 0$$

$$\text{so... } y = -\sqrt{x}$$

$$\boxed{f^{-1}(x) = -\sqrt{x}} \quad D: x \geq 0$$

Note: $g(x)$ is a function
(one-to-one, actually).

Thus $g^{-1}(x)$ has to be
one-to-one since its
inverse ($g(x)$) is a fxn.
 $y = x^2 + 1$ is not one-to-one,
unless domain
is $x \geq 0$.

eg5: Given $f(x) = x^2 + 2$, find $f^{-1}(x)$.

If $f^{-1}(x)$ is NOT a function, restrict the domain of $f(x)$ so that it is.

$$f: f(x) = x^2 + 2 \quad D: x \in \mathbb{R}$$

$$y = x^2 + 2 \quad R: y \geq 2$$

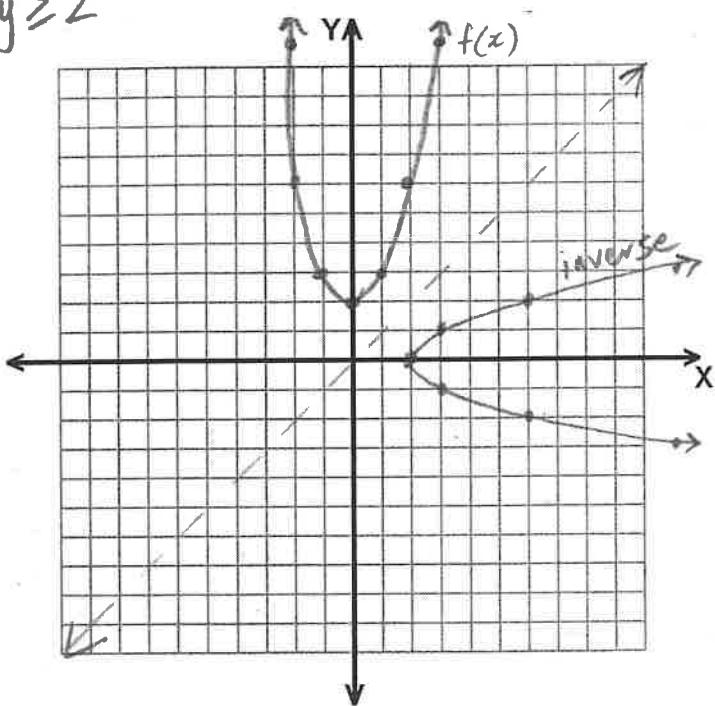
$$f^{-1}: x = y^2 + 2$$

$$y^2 = x - 2$$

$$\sqrt{y^2} = \sqrt{x-2} \quad D: x \geq 2$$

$$y = \pm \sqrt{x-2} \quad R: y \in \mathbb{R}$$

a relation!



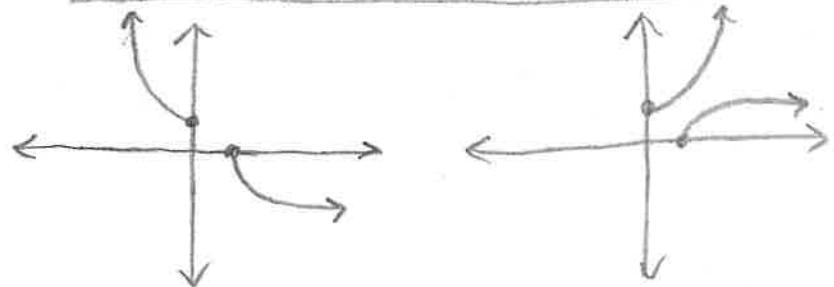
We must restrict f 's domain to allow f^{-1} to be a function (we must restrict the least x -values possible)

f is symmetrical, so restrict HALF of it!

* find x -value of vertex

Possible restrictions:

$$x \leq 0 \quad \text{or} \quad x \geq 0$$



Transformations involving Inverse Functions

- when 'inversing' a function is part of a transformation protocol, order is CRITICAL:

① Inverse!

② Reflections / Expansions - Compressions

③ Translations

Q6: Given point (a, b) on $y = f(x)$, find the point existing on:

a) $y = f^{-1}(x) \rightarrow (b, a)$

b) $y = f^{-1}(x-1) \rightarrow (b, a) \rightarrow (b+1, a)$

c) $y = f^{-1}(x) + 3 \rightarrow (b, a) \rightarrow (b, a+3)$

d) $y = -2f^{-1}(3x) \rightarrow (b, a) \rightarrow (\frac{1}{3}b, -2a)$

e) $y = \frac{1}{2}f^{-1}(-\frac{1}{4}(x+3)) - 1 \rightarrow (b, a) \rightarrow (-4b-3, \frac{1}{2}a-1)$

P. 90
1-12

8 only do 2nd part

Ch. 2.6 - Combined Transformations

$$y = af(b(x-c)) + d$$

a: \rightarrow Vertical expansion/compression
 \rightarrow Vertical reflection

b: \rightarrow Horizontal expansion/compression
 \rightarrow Horizontal reflection

c: \rightarrow Horizontal translation
(* make sure b is factored out of 'internal' binomial!)

d: \rightarrow Vertical translation

'BASIC' FUNCTION : $y = f(x)$.

i.e. When $a, b = \underline{1}$
 $c, d = \underline{0}$

When applying transformations, do so in the following order (unless otherwise stated):

1. Inverse

2. a. Expansion/Compression } * interchangeable
b. Reflection }

3. Translation

eg1: When $y = f(x)$ is transformed to
 $y = -2f(3(x+5)) - 7$,

a) describe, in words, the transformations:

Vertical: * reflect over x -axis
(y) * vertically expand by factor 2
* shift down 7.

Horizontal: * horizontally compress by factor $\frac{1}{3}$
(x) * shift left 5.

b) If $(6, -3)$ is a point on $y = f(x)$,
what point must exist on the transformed
function?

$$6 \times \left(\frac{1}{3}\right) - 5 = -3$$

$$-3 \times (-1) \times (2) - 7 = -1$$

$$\boxed{(-3, -1)}$$

c) If $(2, -5)$ is on the transformed
function, what point must be on $y = f(x)$?

$$x \times \left(\frac{1}{3}\right) - 5 = 2$$

$$(y)(-1)(2) - 7 = -5$$

$$\frac{x}{3} = 7$$

$$-2y = 2$$

$$x = 21$$

$$y = -1$$

$$\boxed{(21, -1)}$$

q2: If $(-5, 2)$ is on $y = f(x)$, what point must be on $y = \frac{f(2x-6) + 2}{2}$?

$$\begin{aligned} y &= \frac{f(2x-6) + 2}{2} \\ &= \frac{1}{2}f(2x-6) + 1 \\ &= \frac{1}{2}f(2(x-3)) + 1 \end{aligned} \quad \left. \begin{array}{l} \underline{x: -5 \times \frac{1}{2} + 3 = \frac{1}{2}} \\ \underline{y: 2 \times \frac{1}{2} + 1 = 2} \end{array} \right\} \boxed{\left(\frac{1}{2}, 2\right)}$$

q3: If the point $(3, 2)$ is on $y = f(x)$, then what point must be on $y = -4f^{-1}(6-3x) + 1$?

$$y = -4f^{-1}(6-3x) + 1$$

$$y = -4f^{-1}(-3x+6) + 1$$

$$y = -4f^{-1}(-3(x-2)) + 1$$

Inverse FIRST! $(3, 2)$ becomes $(2, 3)$

$$\underline{x: 2 \times -\frac{1}{3} + 2 = \frac{4}{3}}$$

$$\underline{y: 3 \times -4 + 1 = -11}$$

'new x' 'new y'

$$\boxed{\left(\frac{4}{3}, -11\right)}$$

Q4: If the point $(-3, 2)$ is on
 $y = -3f(2(x+4)) - 1$, then what
point must be on $y = f^{-1}(x)$?

THINK! 'Backwards' again...

Find point that must be on $y = f(x)$,
then 'inverse' it.

So...

$$(x) \left(\frac{1}{2}\right) - 4 = -3$$

$$\frac{x}{2} = 1$$

$$x = 2$$

$$(y)(-3) - 1 = 2$$

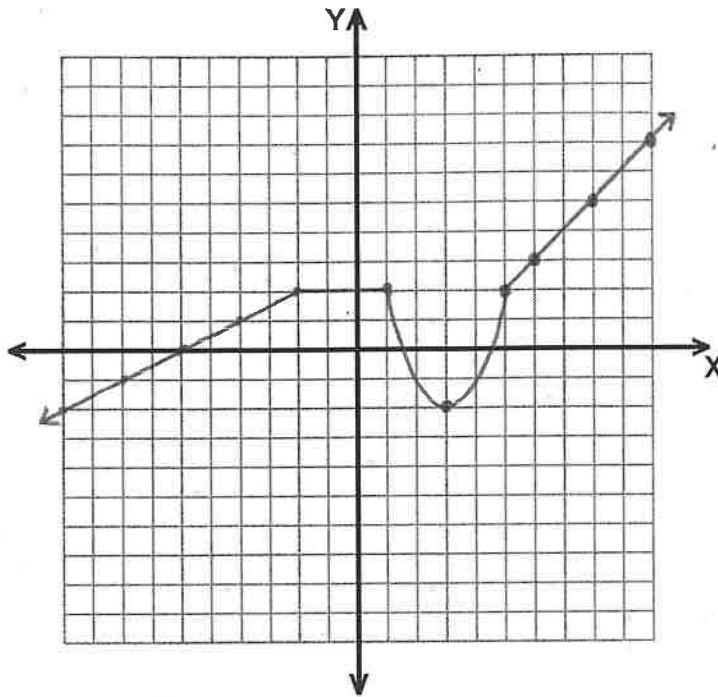
$$-3y = 3$$

$$y = -1$$

INVERSE!

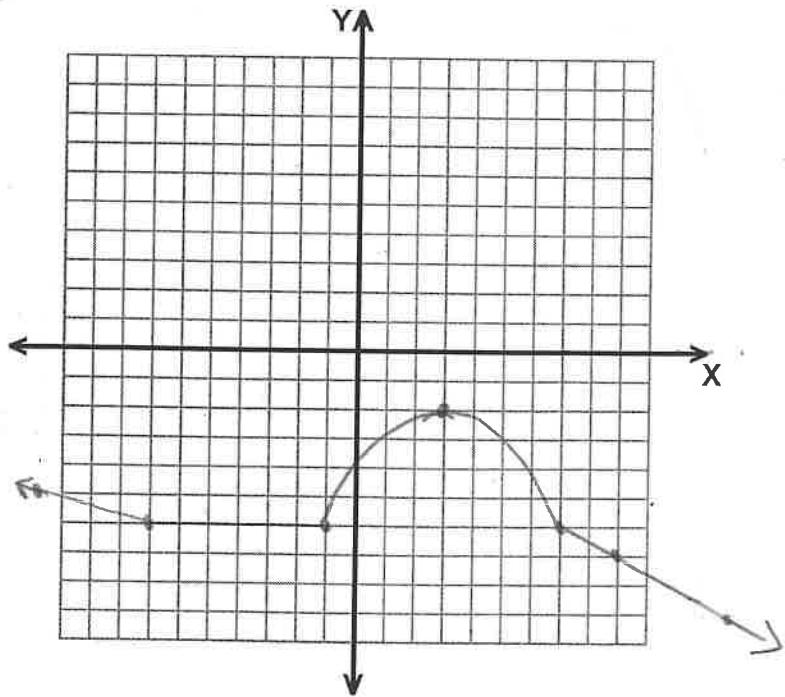
$$\boxed{(-1, 2)}$$

eg 5: Given the graph of $y = f(x)$ below,
sketch graphs of each of the following:



a) $y = -f\left(\frac{1}{2}(x+3)\right) - 4$

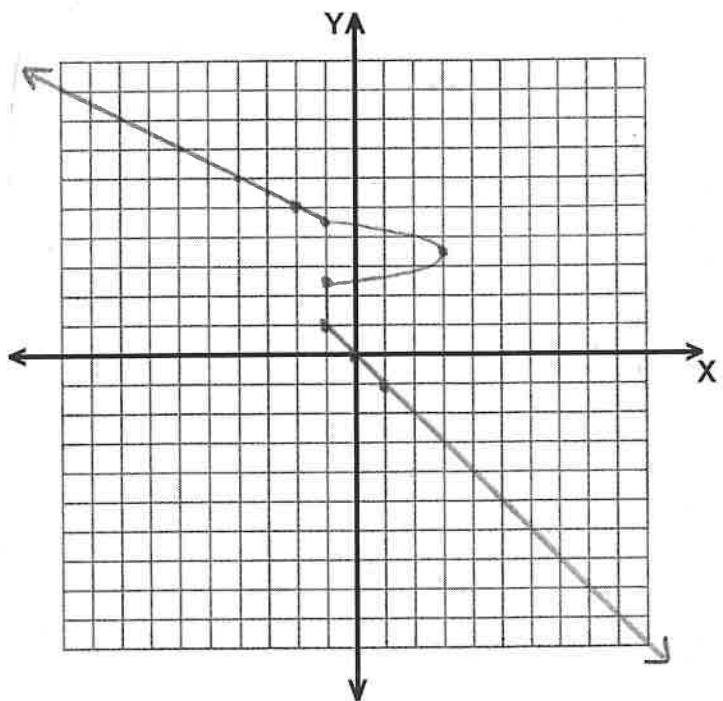
x	y
-15 \leftarrow -12 \leftarrow -6	0 \rightarrow -4
-11 \leftarrow -8 \leftarrow -4	1 \rightarrow -1 \rightarrow -5
-7 \leftarrow -4 \leftarrow -2	2 \rightarrow -2 \rightarrow -6
-1 \leftarrow 2 \leftarrow 1	2 \rightarrow -2 \rightarrow -6
3 \leftarrow 6 \leftarrow 3	-2 \rightarrow 2 \rightarrow -2
7 \leftarrow 10 \leftarrow 5	2 \rightarrow -2 \rightarrow -6
9 \leftarrow 12 \leftarrow 6	3 \rightarrow -3 \rightarrow -7



$$b) \quad y = \frac{1}{2} f^{-1}(-(x-1)) + 2$$

INVERSE FIRST!

x	y
$1 < 0$	$-6 \rightarrow -3 \rightarrow -1$
$0 < -1 < 1$	$-4 \rightarrow -2 \rightarrow 0$
$-1 < -2 < 2$	$-2 \rightarrow -1 \rightarrow 1$
$-1 < -2 < 2$	$1 \rightarrow \frac{1}{2} \rightarrow \frac{5}{2}$
$3 < 2 < -2$	$3 \rightarrow \frac{3}{2} \rightarrow \frac{7}{2}$
$-1 < -2 < 2$	$5 \rightarrow \frac{5}{2} \rightarrow \frac{1}{2}$
$-2 < -3 < 3$	$6 \rightarrow 3 \rightarrow 5$



p. 97

1, 2, 3, 5ab, 6, 7. typo: p 412 7f } graphing error

+

p. 100 Chapter Review (omit # 4, 21, 37, 63)