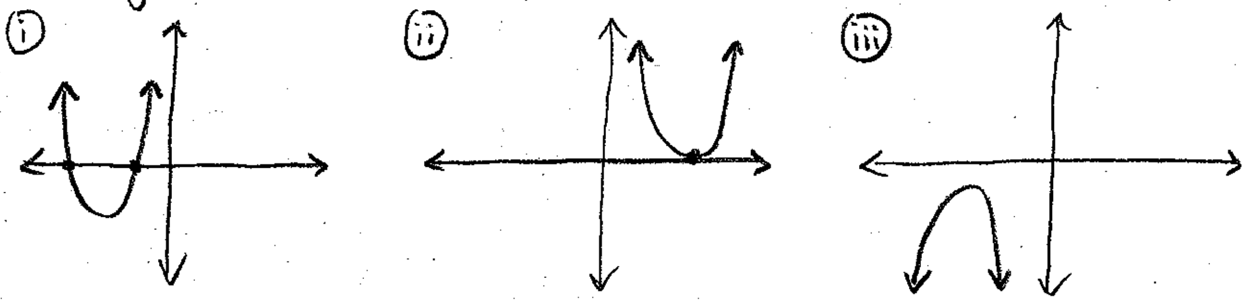
Ch. 5.1 - Properties of Quadratic Functions

- a QUADRATIC FUNCTION is a 2^{nd} degree polynomial function (ie. contains a x^2 term).

eg: $y = x^2$; $y = (x-2)(x-3)$; $y = x^2 + 6x + 8$

- the graph of a quadratic fcn is a PARABOLA.

eg: (i) (ii) (iii)



- two x-intercepts — one x-int. — no x-ints.

- GENERAL FORM quadratic fcn: $f(x)$ OR y $= ax^2 + bx + c$

a poor form for graphing parabolas (but useful in introducing quadratics).

where a, b, and c are real numbers and $a \neq 0$.

If $a > 0$, then parabola opens UPWARDS.
* see (i) above

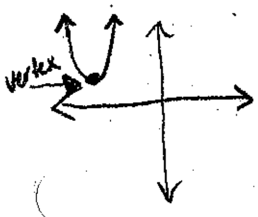
If $a < 0$, then parabola opens DOWNWARDS.
* see (iii) above

The y-intercept - the point where the parabola intersects the y-axis.
- it is found by setting $x = 0$.
- there is always ONE y-intercept, the point $(0, c)$.

The x-intercept(s) - the point(s) where the parabola intersects the x-axis.

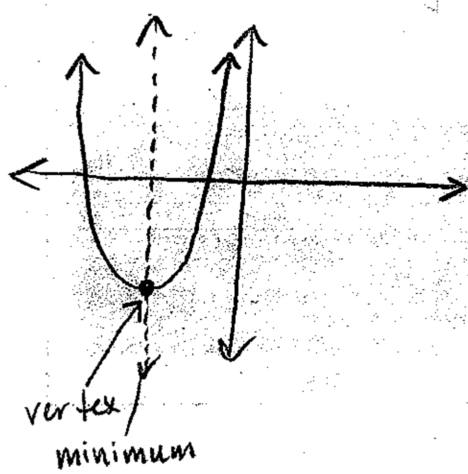
- there can be 0, 1, or 2 x-intercepts, depending on the quadratic fxn.
- see (i), (ii), and (iii) above.
- found by setting $y = 0$ (or $f(x) = 0$) then solving the quadratic equation

The Vertex - the maximum or minimum point of a parabola. The y-value of the vertex is the min. or max. value of the parabola.



Axis of Symmetry - the VERTICAL LINE that passes through the VERTEX of the parabola.

- the equation of the axis of symmetry is $x = n$, where n is the x-value of the vertex.



* also found by finding the midpoint between the two x-intercepts (or any other symmetrical points → points sharing the same y-value).

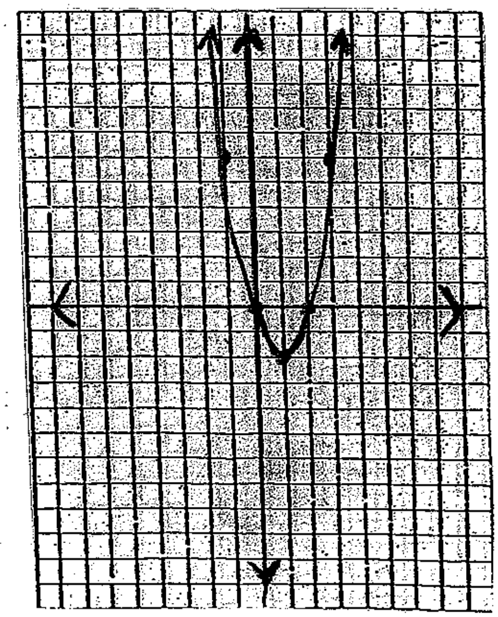
Domain and Range

- the Domain of all quadratic fxns ($y = ax^2 + bx + c$) is that $x \in \mathbb{R}$ (all real #s) since there is no limitation on what x can be.
- the Range can be deduced from the vertex.
  - if the vertex is (x, n), then the range is:

$y \geq n$, when $a > 1$ (opens UP).

$y \leq n$, when $a < 1$ (opens DOWN).

eg1: Given the graph of the fxn $y = 2x^2 - 4x$, determine the:



a) Vertex $(1, -2)$

b) Axis of Symmetry $x = 1$

c) y-intercept $(0, 0)$

d) x-intercept(s) $(0, 0)$ $(2, 0)$

e) Domain/Range $x \in \mathbb{R}$
$y \geq -2$

f) Minimum value $y = -2$

eg2: Given the function $f(x) = x^2 + 2x - 15$, find the:

a) x-intercept(s)

$$0 = x^2 + 2x - 15$$
$$0 = (x+5)(x-3)$$
$$x = -5, 3 \qquad \boxed{(-5,0)\ (3,0)}$$

b) y-intercept

$$y = (0)^2 + 2(0) - 15$$
$$y = 15 \qquad \boxed{(0,15)}$$

c) Equation of the axis of symmetry

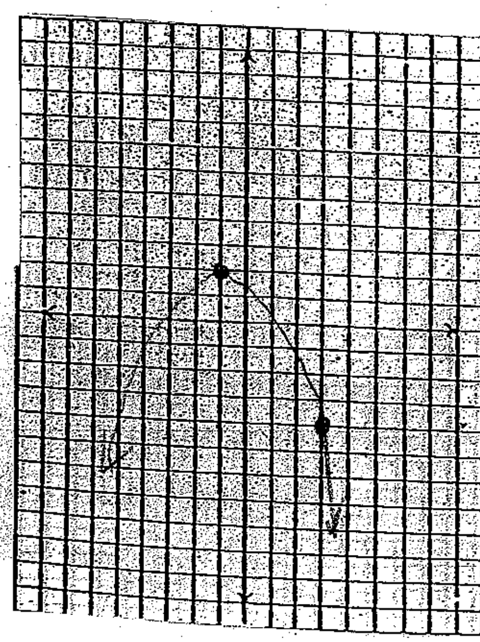
x-ints $= (-5,0)\ (3,0)$

$\frac{1}{2}$-way is -1 $\qquad \boxed{x = -1}$

eg3: Find a point on a quadratic fxn that has vertex $(-1,2)$ and passes through the point $(3,-4)$.

HINT: a graph will help!

opens down!



Using symmetry...

another point is

$(x, -4)$

$(-5, -4)$

p. 220-223 # 1-7.

Ch. 5.2 - The Standard Form of a Quadratic Fxn

General form: $y = ax^2 + bx + c$

Info this provides: (i) Direction of opening $\Rightarrow$ value of $\underline{a}$.

(ii) y-intercept $\Rightarrow$ value of $\underline{c}$.

* biggest problem? no info about VERTEX.

Standard form: y or $f(x) = \underline{a(x-h)^2 + k}$

read as opposite (h) — read as is (k)

$h =$ horizontal shift (x)

$q =$ vertical shift (y)

$(h, k) =$ VERTEX

$a =$ Vertical stretch

* Equation of Axis of Symmetry will always be: $x = \underline{h}$

HINTS for parabola's shape:

- If $|a| > 1 \Rightarrow$ parabola narrows horizontally (stretched vertically)
- If $0 < |a| < 1 \Rightarrow$ parabola widens horizontally (shrunk vertically)
- If $|a| = 1 \Rightarrow$ parabola "normal"

To graph a parabola:

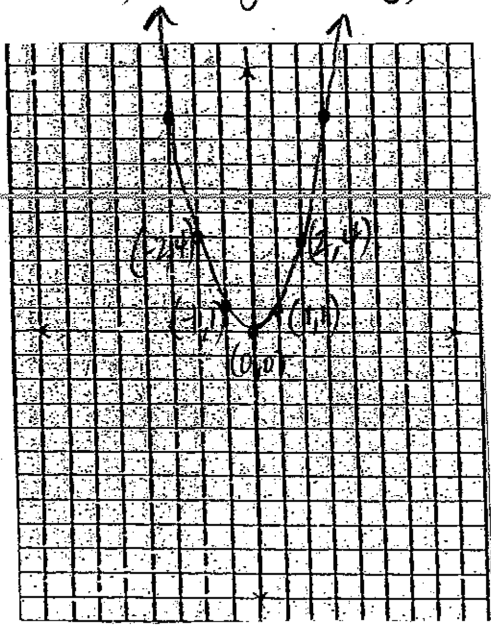
(1) Determine and plot the VERTEX.

(2) Then graph $\underline{y = ax^2}$ from the vertex.

* remember, parabolas are SYMMETRICAL!

eg1: Graph $y = x^2$ and find:

a) VERTEX ; b) Equation of the Axis of Symmetry ;
c) y-intercept ; d) x-intercept(s) ; e) Domain ;
f) Range ; g) Maximum/minimum value.



$y = 1(x-0)^2 + 0$

a) $(0,0)$ then graph $y = 1x^2$ from vertex.

b) $x = 0$ c) set $x = 0$: $y = 0^2 = \underline{\underline{0}}$ $\boxed{(0,0)}$

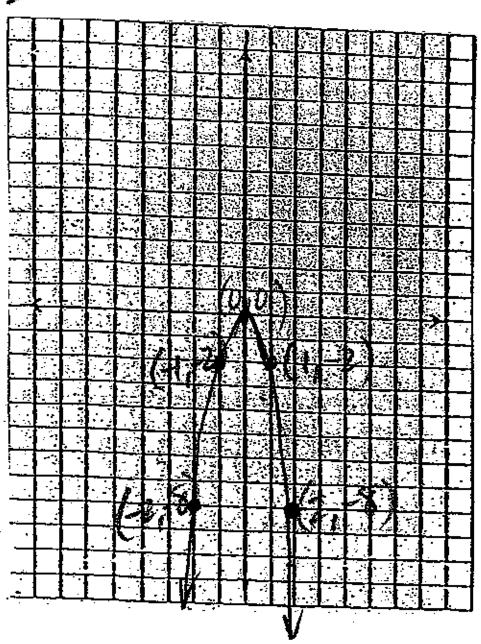
d) set $y = 0$: $\sqrt{0} = \sqrt{x^2}$
$x = 0$ $\boxed{(0,0)}$

e) D: $x \in \mathbb{R}$ f) R: $y \geq 0$

g) Min @ $y = 0$.

eg2: Graph $y = -2x^2$ and find a-g (above):

$y = -2(x-0)^2 + 0$



a) $(0,0)$; then graph $y = -2x^2$ from vertex

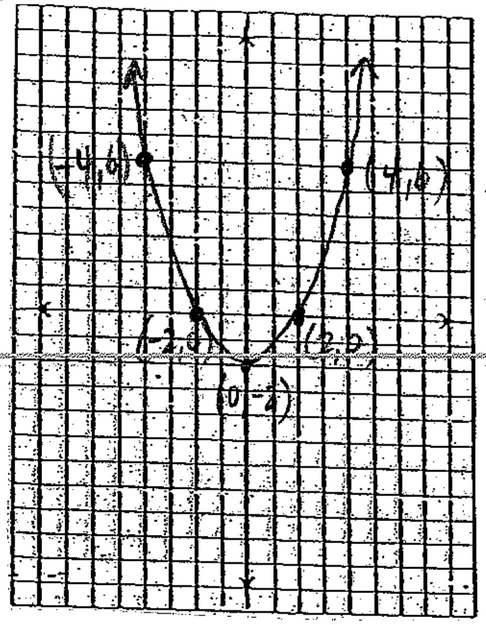
b) $x = 0$ c) set $x = 0$: $y = -2(0)^2 = 0$ $\boxed{(0,0)}$

d) set $y = 0$: $0 = -2x^2$
$\sqrt{0} = \sqrt{x^2} \Rightarrow x = 0$ $\boxed{(0,0)}$

e) D: $x \in \mathbb{R}$ f) R: $y \leq 0$

g) Max @ $y = 0$

eg3: Graph $y = \frac{1}{2}x^2 - 2$; find a–g:



$y = \frac{1}{2}(x-0)^2 - 2$

a) $V = (0,-2)$; then graph $y = \frac{1}{2}x^2$ from vertex

b) $x = 0$ c) set $x = 0$: $y = \frac{1}{2}(0)^2 - 2 = -2$ $\boxed{(0,-2)}$

d) set $y = 0$: $0 = \frac{1}{2}x^2 - 2$

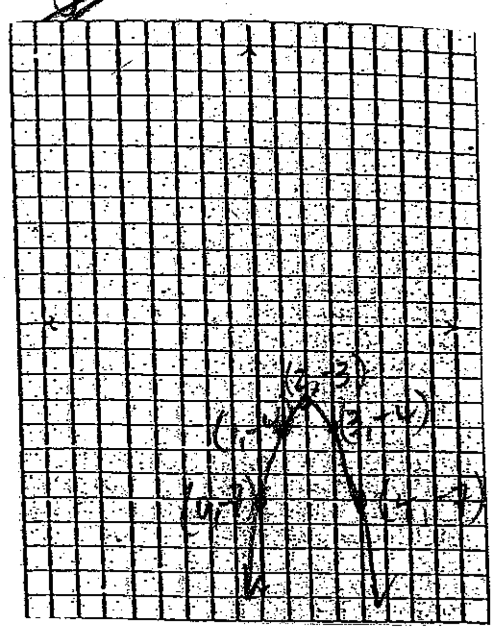
$2 = \frac{1}{2}x^2$

$x^2 = 4 \Rightarrow x = \pm 2$ $\boxed{(2,0) \quad (-2,0)}$

e) D: $x \in \mathbb{R}$ f) R: $y \geq -2$

g) Min @ $y = -2$

eg4: Graph $y = -(x-2)^2 - 3$; find a–g:



$y = -1(x-2)^2 - 3$

a) $V = (2,-3)$; then graph $y = -1x^2$ fr. vertex

b) $x = 2$

c) set $x = 0$: $y = -(0-2)^2 - 3$

$y = -4 - 3 = -7$ $\boxed{(0,-7)}$

d) set $y = 0$: $0 = -(x-2)^2 - 3$

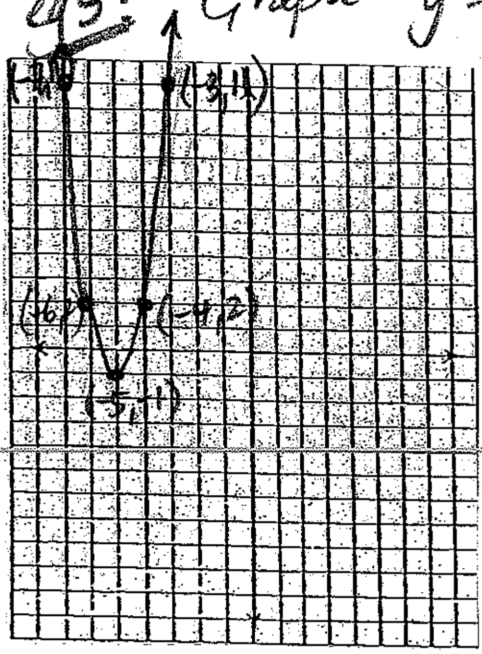
$\sqrt{(x-2)^2} = \sqrt{-3}$

no x-intercepts!

e) D: $x \in \mathbb{R}$ f) R: $y \leq -3$

g) Max @ $y = -3$

eg5: Graph $y = 3(x+5)^2 - 1$; find a–g:



a) $V = (-5,-1)$; then graph $y = 3x^2$

b) $x = -5$

c) set $x = 0$: $y = 3(0+5)^2 - 1$

$y = 75 - 1 = 74$ $\boxed{(0,74)}$

d) set $y = 0$:

$0 = 3(x+5)^2 - 1$

$1 = 3(x+5)^2$

$\frac{1}{3} = (x+5)^2$

$x+5 = \frac{\pm 1}{\sqrt{3}}$

$x+5 = \frac{\pm\sqrt{3}}{3}$

$x = \frac{\pm\sqrt{3}}{3} - 5$

$x = \frac{\pm\sqrt{3}}{3} - \frac{15}{3}$

$x = \boxed{\frac{\pm\sqrt{3} - 15}{3}}$

$\boxed{\left(\frac{\sqrt{3}-15}{3}, 0\right) \quad \left(\frac{-\sqrt{3}-15}{3}, 0\right)}$

e) D: $x \in \mathbb{R}$

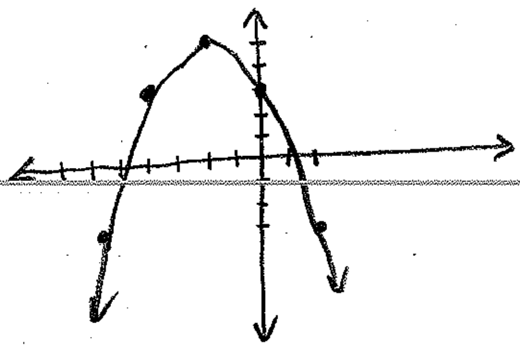
f) R: $y \geq -1$

g) Min. @ $y = -1$

Hwk. p. 227 – 230 # 1 – 7.

Ch. 5.3 - Finding the Equation of a Parabola

eg1: Given the following graph, we can determine the parabola's equation (function):



Vertex? $(-2, 5)$

Other known points? $(0,3), (-4,3), (2,-3), (-6,-3)$

Need to find a: $y = a(x-h)^2 + k$

Remember: $V = (h, k)$ and any of the above 4 points can serve as (x, y)

we'll use $(0,3)$:

$$3 = a(0-(-2))^2 + 5$$

$$3 = a(0+2)^2 + 5$$

$$3 = 4a + 5$$

$$-2 = 4a$$

$$a = -\frac{1}{2}$$

So... parabola's equation is:

$$\boxed{y = -\frac{1}{2}(x+2)^2 + 5}$$

eg2: Find the equation of the parabola function whose graph has a vertex of $(4,8)$ and an x-intercept of 6.

If x-int. is 6, then point is $(6,0)$.

Vertex $(4,8)$

$$y = a(x-h)^2 + k$$
$$0 = a(6-4)^2 + 8$$
$$0 = 4a + 8$$
$$-8 = 4a$$
$$a = -2$$

$$\boxed{\begin{matrix} y \\ \text{or} \\ f(x) \end{matrix} = -2(x-4)^2 + 8}$$

eg3: Find the equation of a quadratic fxn with points $(-1,0)$, $(0,\frac{3}{2})$, $(3,0)$.

$$y = a(x-h)^2 + K$$

$(-1,0)$ and $(3,0)$ are symmetric so... the x-value of the vertex must be the midpt. of -1 and $3 \Rightarrow 1$

$$y = a(x-1)^2 + k$$ plug in a point $(-1,0)$ say

$$0 = a(-1-1)^2 + k$$
$$0 = 4a + k$$
$k = -4a$ now, use a 2nd point (must be the non-symmetrical one) $(0,\frac{3}{2})$

$$\frac{3}{2} = a(0-1)^2 - 4a$$
$$\frac{3}{2} = -3a$$
$$3 = -6a$$
$$a = -\frac{1}{2}$$
$$k = 2$$

$$\boxed{y = -\frac{1}{2}(x-1)^2 + 2}$$

eg4: Find an equation of a quadratic function with points $(3,-4)$, $(-3,2)$, $(1,2)$.

Two symmetrical points $\Rightarrow$ $(-3,2)$ + $(1,2)$

x-value of vertex = $\boxed{-1}$

$y = a(x+1)^2 + k$ plug in $(1,2)$

$2 = a(1+1)^2 + k$

$2 = 4a + k$

$k = 2 - 4a$

Plug in non-symmetrical point $(3,-4)$

$-4 = a(3+1)^2 + (2-4a)$

$-4 = 16a + 2 - 4a$

$-6 = 12a$

$a = -\frac{1}{2}$ (again) $\therefore k = 2 - (4(-\frac{1}{2})) = \underline{4}$

$$\boxed{y = -\frac{1}{2}(x+1)^2 + 4}$$

p. 233 - 236 #1-9

Ch. 5.4 - Converting General to Standard Form

Remember:

General Form: $\underline{y = ax^2 + bx + c}$ $\boxed{\text{no info about vertex}}$

Standard Form: $\underline{y = a(x-h)^2 + k}$ <u>info about vertex.</u>

How to convert? <u>Complete the Square!</u>

eg1: Convert $y = x^2 - 6x$ to standard form, then graph. $\boxed{a=1}$

$\boxed{-3}$ 9

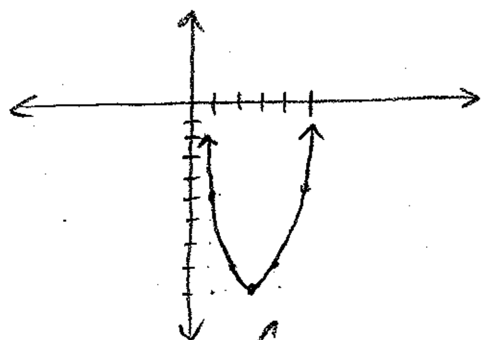
$y = x^2 - 6x$

$y + 9 = x^2 - 6x + 9$

$y + 9 = (x-3)^2$

$y = (x-3)^2 - 9$

Vertex = $(3, -9)$



eg2: Find the vertex and x-intercepts of $f(x) = x^2 + 8x + 10$.

$y = x^2 + 8x + 10$

$\boxed{+4}$ 16

$y - 10 = x^2 + 8x$

$y - 10 + 16 = x^2 + 8x + 16$

$y + 6 = (x+4)^2$

$y = (x+4)^2 - 6$

$\boxed{V = (-4, -6)}$

x-ints: set $y = 0$: $0 = (x+4)^2 - 6$

$\sqrt{6} = \sqrt{(x+4)^2}$

$x + 4 = \pm\sqrt{6}$

$x = \pm\sqrt{6} - 4$

$\boxed{(\sqrt{6} - 4, 0)\ (-\sqrt{6} - 4, 0)}$

eg3: Find the vertex and x-intercepts of:

$y = -2x^2 + 4x - 5$

$y + 5 = -2x^2 + 4x$ $\boxed{-1}$ 1

$y + 5 = -2(x^2 - 2x)$

$y + 5 - 2 = -2(x^2 - 2x + 1)$

$y + 3 = -2(x - 1)^2$

$y = -2(x-1)^2 - 3$

$\boxed{V = (1, -3)}$

x-ints: set $y = 0$

$0 = -2(x-1)^2 - 3$

$2(x-1)^2 = -3$

$\sqrt{(x-1)^2} = \sqrt{-\frac{3}{2}}$

$\boxed{\text{no } x\text{-ints}}$

eg4: Find the vertex of $y = 5x - 3x^2$

$y = -3x^2 + 5x$ $\boxed{-\frac{5}{6}}$ $\frac{25}{36}$

$y = -3(x^2 - \frac{5}{3}x)$

$y - \frac{75}{36} = -3(x^2 - \frac{5}{3}x + \frac{25}{36})$

$y - \frac{25}{12} = -3(x - \frac{5}{6})^2$

$y = -3(x - \frac{5}{6})^2 + \frac{25}{12}$

$V = (\frac{5}{6}, \frac{25}{12})$

p. 239-244 # 4-14 (omit 6i, j)

and + QUIZ

p. 247-248 # 1-6 (do not use vertex formula)

omit 1j

Solutions to Qs 7-14 p. 244

7) $y = x^2 + kx + 4$

$y - 4 = x^2 + kx$ $\quad \boxed{\frac{k}{2}}$ $\frac{k^2}{4}$

$y - 4 + \frac{k^2}{4} = x^2 + kx + \frac{k^2}{4}$

$y + \frac{k^2}{4} - 4 = \left(x + \frac{k}{2}\right)^2$

$y = \left(x + \frac{k}{2}\right)^2 + 4 - \frac{k^2}{4}$

$\boxed{V = \left(-\frac{k}{2},\ 4 - \frac{k^2}{4}\right)}$

8) $y = 2x^2 + kx + k^2$ $\quad \boxed{\frac{k}{4}}$ $\frac{k^2}{16}$

$y - k^2 = 2\left(x^2 + \frac{k}{2}x\right)$

$y - k^2 + \frac{2k^2}{16} = 2\left(x^2 + \frac{k}{2}x + \frac{k^2}{16}\right)$

$y - \frac{14k^2}{16} = 2\left(x + \frac{k}{4}\right)^2$

$y = 2\left(x + \frac{k}{4}\right)^2 + \frac{7k^2}{8}$

$\boxed{V = \left(-\frac{k}{4},\ \frac{7k^2}{8}\right)}$

9) $y = 2x^2 + ax + b^2$

$y - b^2 = 2\left(x^2 + \frac{a}{2}x\right)$ $\quad \boxed{\frac{a}{4}}$ $\frac{a^2}{16}$

$y - b^2 + \frac{a^2}{8} = 2\left(x^2 + \frac{a}{2}x + \frac{a^2}{16}\right)$

$y = 2\left(x + \frac{a}{4}\right)^2 + b^2 - \frac{a^2}{8}$

$\boxed{V = \left(-\frac{a}{4},\ b^2 - \frac{a^2}{8}\right)}$

10) $y = px^2 - 3x + p$

$y - p = p\left(x^2 - \frac{3}{p}x\right)$ $\quad \boxed{\frac{-3}{2p}}$ $\frac{9}{4p^2}$

$y - p + \frac{9}{4p} = p\left(x^2 - \frac{3}{p}x + \frac{9}{4p^2}\right)$

$y + \frac{9 - 4p^2}{4p} = p\left(x - \frac{3}{2p}\right)^2$

$y = p\left(x - \frac{3}{2p}\right)^2 + \frac{4p^2 - 9}{4p}$

$\boxed{V = \left(\frac{3}{2p},\ \frac{4p^2 - 9}{4p}\right)}$

11) $y = kx(8 - x)$

$y = -kx^2 + 8kx$ $\quad \boxed{-4}$ 16

$y = -k\left(x^2 - 8x\right)$

$y - 16k = -k\left(x^2 - 8x + 16\right)$

$y = -k(x - 4)^2 + 16k$

$\boxed{V = (4,\ 16k)}$

12) $y = ax^2 + 4x - 4$ Vertex x-value $= \underline{\underline{6}}$

$$y + 4 = a\left(x^2 + \frac{4}{a}x\right) \quad \boxed{\frac{2}{a}} \quad \frac{4}{a^2}$$

$$y + 4 + \frac{4}{a} = a\left(x + \frac{4}{a}x + \frac{4}{a^2}\right)$$

$$y = a\left(x + \frac{2}{a}\right)^2 - \frac{4 + 4a}{a}$$

$$y = a\left(x - \left(-\frac{2}{a}\right)\right)^2 - \frac{4 + 4a}{a}$$

$$-\frac{2}{a} = 6$$

$$\boxed{a = -\frac{1}{3}}$$

13) $y = 2z^2 + bx - 3$ y-value of vertex $= -5$

$$y + 3 = 2\left(x^2 + \frac{b}{2}x\right) \quad \boxed{\frac{b}{4}} \quad \frac{b^2}{16}$$

$$y + 3 + \frac{b^2}{8} = 2\left(x^2 + \frac{b}{2}x + \frac{b^2}{16}\right)$$

$$y = 2\left(x + \frac{b}{4}\right)^2 - \left(3 + \frac{b^2}{8}\right)$$

$$-\left(3 + \frac{b^2}{8}\right) = -5$$

$$3 + \frac{b^2}{8} = 5$$

$$\frac{b^2}{8} = 2$$

$$b^2 = 16$$

$$\boxed{b = \pm 4}$$

14) $y = 0.1x^2 + 7x + c$ y-value of vertex = -120.5

$y - c = 0.1(x^2 + 70x)$ $\boxed{35}$ 1225

$y - c + 122.5 = 0.1(x^2 + 70x + 1225)$

$y = 0.1(x+35)^2 + c - 122.5$

$c - 122.5 = -120.5$

$\boxed{c = 2.0}$

Chapter 5.6 – Applications of Quadratic Functions (Word Problems)

eg1: Mary stands on the top of a building and throws a ball upwards. The ball travels according to the equation $h = -16t^2 + 384t + 50$, where h is the height of the ball off the ground in meters at t seconds after it was thrown.
a. How far is Mary above the ground when she throws the ball?
b. What is the highest vertical point that the ball reaches?
c. How long does it take for the ball to reach this highest vertical point?
d. After how many seconds does the ball hit the ground?

$h = -16t^2 + 384t + 50$

$h - 50 = -16(t^2 - 24t)$ $\boxed{-12}$ 144

$h - 50 - 2304 = -16(t^2 - 24t + 144)$

$h - 2354 = -16(t-12)^2$

$h = -16(t-12)^2 + 2354$

a) when $t = 0 \Rightarrow h = 50$ m

b) max. $h = 2354$ m (superwoman)

c) @ max height, time = 12 s.

d) set $h = 0$

$0 = -16(t-12)^2 + 2354$

$\frac{-2354}{-16} = (t-12)^2$

$147.125 = (t-12)^2$

$\pm 12.13 = t - 12$

$\boxed{t = 24.13 \text{ s}}$

eg2: Find two numbers whose difference is 12 and whose product is a minimum. What is the minimum product?

Let x = one number

Let y = the other

$x - y = 12$

$y = x - 12$

$P = xy$

$P = (x)(x-12)$

$P = x^2 - 12x$ $\boxed{-6}$ 36

$P + 36 = x^2 - 12x + 36$

$P = (x-6)^2 - 36$

$$\boxed{x = 6, \quad y = -6 \quad P = -36}$$

eg3: Two numbers have a sum of 20. Does the sum of their squares have a *maximum* or a *minimum* value? Determine this value and the two numbers.

Let x = one number

Let y = the other

$x + y = 20$

$y = 20 - x$

$S = x^2 + y^2$

$S = x^2 + (20-x)^2$

$S = 2x^2 - 40x + 400$ $\boxed{-10}$ 100

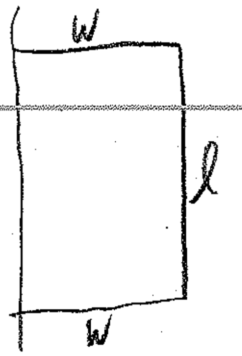
$S - 400 = 2(x^2 - 20x)$

$S - 400 + 200 = 2(x^2 - 20x + 100)$

$S = 2(x-10)^2 + 200$

$$\boxed{x = 10; \quad y = 10 \quad S = 200}$$

eg4: A rancher has 800m of fencing to enclose a rectangular cattle pen along a riverbank. No fencing is required along the riverbank. Find the dimensions that would maximize the area for the cattle to graze. Also, what is the max. area?



$$P = w + l + w$$

$$800 = 2w + l$$

$$l = 800 - 2w$$

$$A = lw$$

$$A = (800-2w)(w)$$

$$A = -2w^2 + 800w \qquad \boxed{-200} \quad 40000$$

$$A = -2(w^2 - 400w)$$

$$A - 80000 = -2(w^2 - 400w + 40000)$$

$$A = -2(w-200)^2 + 80000$$

$$\boxed{\begin{aligned} W &= 200 \\ l &= 400 \\ A &= 80000 \end{aligned}}$$

eg5: A 400-room hotel is three-quarters full when the room rate is an average of $80 per night. A survey shows that each $5.00 increase in room rental fee will result in 10 fewer customers. Find the nightly rate that will maximize income. What is the maximum income?

Revenue = (Rate) × (# of ppl.)

Let x = # of $5 increases

$400 \cdot \frac{3}{4} = 300$

$$R = (80 + 5x)(300 - 10x)$$

$$R = -50x^2 + 700x + 24000$$

$$R - 24000 = -50(x^2 - 14x) \quad \boxed{7} \quad 49$$

$$R - 24000 - 2450 = -50(x^2 - 14x + 49)$$

$$R = -50(x-7)^2 + 26450$$

$x = 7$ so... 7 $5 increases = $35 + $80

$\boxed{= \$110}$

Max income = $\boxed{\$26450}$

Homework: p. 256-258 #1-18

Chapter Review: p259-262 #1-13