

Ch. 3.1 - Polynomial Functions

A POLYNOMIAL FUNCTION is of the form:

$$y = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0,$$

Where: $\rightarrow a_n, a_{n-1}, \dots$ are REAL number coefficients (that may be equal to each other).

$$\rightarrow a_n \neq 0$$

$\rightarrow n$ is a WHOLE number ≥ 0 .

$$\rightarrow a_1 x = \underline{a_1 x^1} \text{ and } a_0 = \underline{a_0 x^0}$$

\rightarrow the polynomial is described as being of degree n , and a_n is the LEADING COEFFICIENT (ie. the coefficient of the highest-degree term).

* a polynomial is said to be in STANDARD FORM when it is written in descending order (highest to lowest) of exponents.

eg1: Fill in the table:

POLYNOMIAL IN STANDARD FORM	DEGREE	LEADING COEFF.
$f(x) = -x^4 + 3x^2 + 2x - 5$	4 (QUARTIC)	-1
$g(x) = 2x^3 - x^2 + 4x - 1$	3 (CUBIC)	2
$y = \sqrt{3}x^2 + 2$	2 (QUADRATIC)	$\sqrt{3}$
$k(x) = -4x + 2$	1 (LINEAR)	-4
$j(x) = 3$	0 (CONSTANT)	3

eg 2: Why are each of the following not considered to be polynomial functions?

a) $f(x) = 3x^{-2} + 2x + 5$

* -2 is NOT a whole number ≥ 0 .

b) $g(x) = \sqrt{2}x^3 + \sqrt{-3}x + 1$

* $\sqrt{-3}$ is NOT a REAL number.

c) $h(x) = \frac{2x^2 - 3}{x}$

* $h(x) = \frac{2x^2}{x} - \frac{3}{x} = 2x - \frac{3}{x} = 2x - 3x^{-1}$

d) $j(x) = 3x - 5\sqrt{x}$

* $j(x) = 3x - 5x^{\frac{1}{2}} \rightarrow \frac{1}{2}$ not a whole #.

Graphs of Polynomial Functions

- they are CONTINUOUS with smooth curves.

ie. they can be drawn without lifting your pencil; they have no 'corners'!

- see top of p. 114 for examples and counterexamples.

End Behaviour of Polynomials

$$f(x) = x^n \quad \text{and} \quad f(x) = -x^n,$$

(* each known as a MONOMIAL)

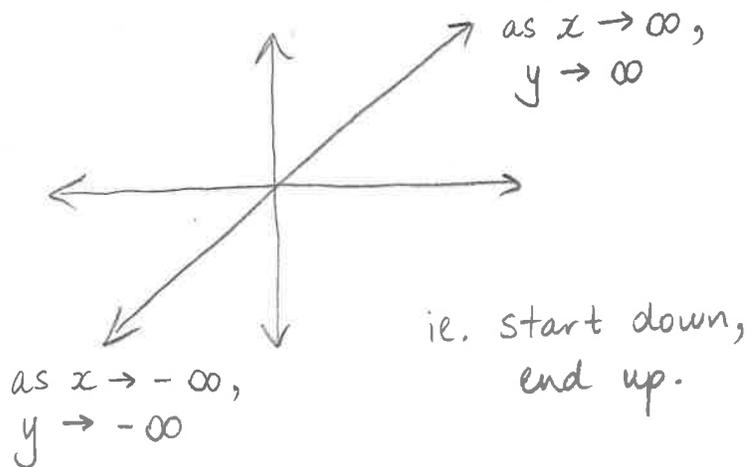
where $n \geq 1$ (n a whole number).

End Behaviour - a function's y -value for 'extreme' negative and positive x -values.

ie. a function's output as $x \rightarrow \pm \infty$.

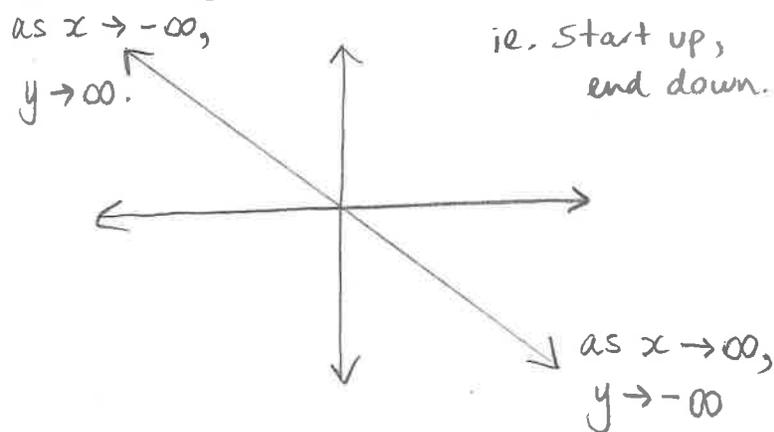
$$f(x) = x$$

$$y = x$$

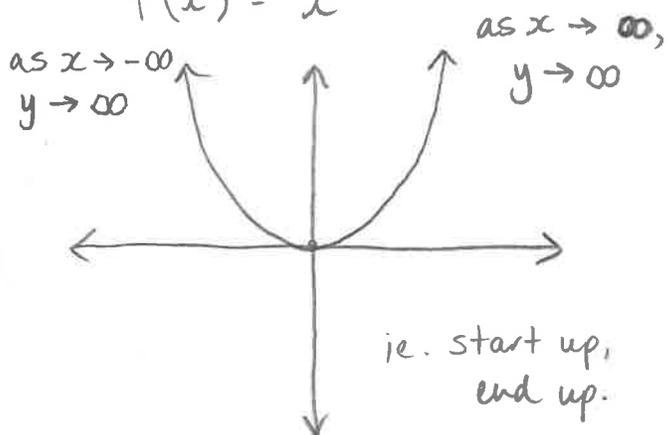


$$f(x) = -x$$

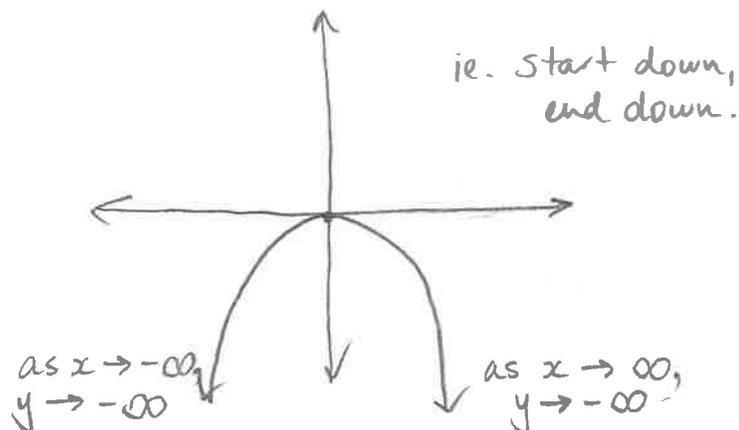
$$y = -x$$

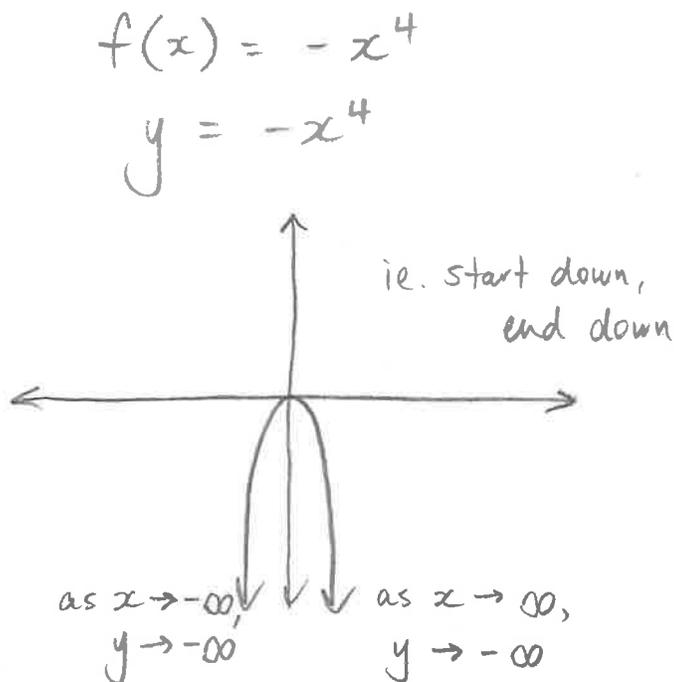
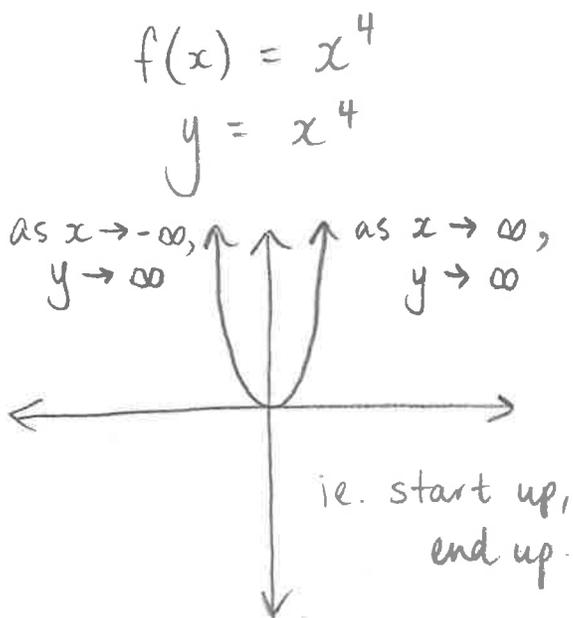
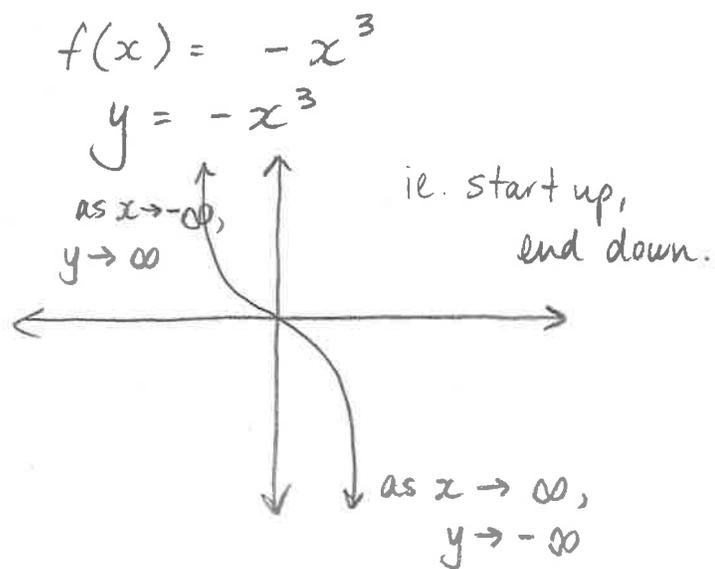
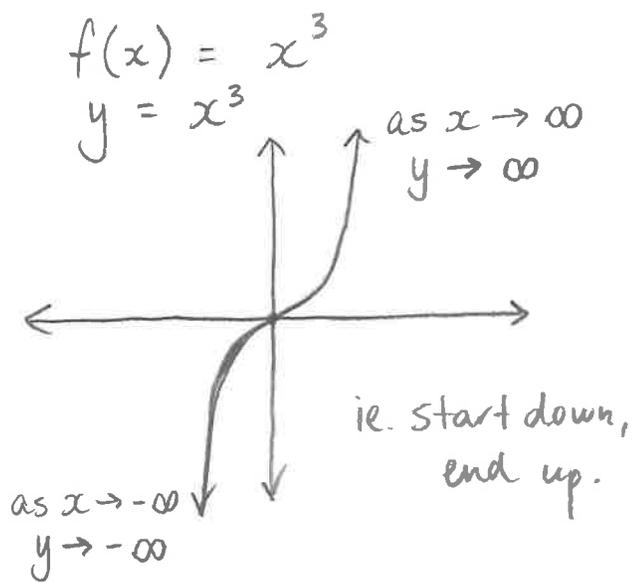


$$f(x) = x^2$$



$$f(x) = -x^2$$





* even though a polynomial may possess more than one term, end behaviour is ONLY defined by a polynomial's leading coefficient (+ or -) and its degree (even or odd).

SUMMARY: Polynomial End Behaviour

DEGREE LEADING COEFFICIENT	EVEN	ODD
POSITIVE	Start UP, end UP.	Start DOWN, end UP.
NEGATIVE	Start DOWN, end DOWN.	Start UP, end DOWN.

Constant Value of a Polynomial Function

If $y = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$,

a_0 represents the CONSTANT value of the polynomial.

Also, a_0 represents the y-intercept of the graph of the polynomial.

ie. the y-value of the function when $x=0$.

eg 3: Find the y-intercept of each of the following polynomial functions:

a) $y = 2x^3 - 6x^2 + x - 4$

* written in STANDARD form, so

$a_0 = -4$, and $y = -4$ $\boxed{(0, -4)}$

OR: $y = 2(0)^3 - 6(0)^2 + (0) - 4$

$y = -4$ $\boxed{(0, -4)}$

$$b) f(x) = -2(x+3)^3$$

* not in STANDARD form!

$$\downarrow$$
$$f(x) = -2(x^2 + 6x + 9)(x+3)$$

$$f(x) = -2(x^3 + 9x^2 + 27x + 27)$$

$$f(x) = -2x^3 - 18x^2 - 54x - 54$$

$$\rightarrow f(0) = -2(0+3)^3$$

$$= -2(27)$$

$$= -54$$

$$\boxed{(0, -54)}$$

$$a_0 = -54$$

$$y\text{-int} = \boxed{(0, -54)}$$

x-INTERCEPTS OF A POLYNOMIAL FXN.

- an x-intercept is also known as a ROOT or a ZERO of a function.

ie. found when we set $y = 0$

eg4: Find the real zero(s) of each:

$$a) f(x) = x(x-4)(2x+1)(x+1)$$

$$\text{set } f(x) \text{ or } y = 0$$

$$0 = x(x-4)(2x+1)(x+1)$$

$$\boxed{x = 0, 4, -\frac{1}{2}, -1}$$

$$b) y = -3x^4 + 3x^2$$

$$0 = -3x^4 + 3x^2$$

$$0 = -3x^2(x^2 - 1)$$

$$0 = -3x^2(x+1)(x-1)$$

$$x = 0, 0, -1, 1$$

$$\boxed{x = 0, \pm 1}$$

$$c) y = x^3 + x$$

$$0 = x(x^2 + 1)$$

$$\boxed{x = 0}$$

* $x^2 + 1 = 0$ has NO solution.

$$d) f(x) = x^3 - 2x^2 - x + 2$$

$$0 = (x^3 - 2x^2)(-x + 2)$$

$$0 = x^2(x-2) - 1(x-2)$$

$$0 = (x-2)(x^2 - 1)$$

$$0 = (x-2)(x+1)(x-1)$$

$$\boxed{x = 2, \pm 1}$$

Notes:

- a polynomial function of degree n has, at most, n real zeros.
- if n is even, there could be 0 to n roots;
- if n is odd, there could be 1 to n roots.

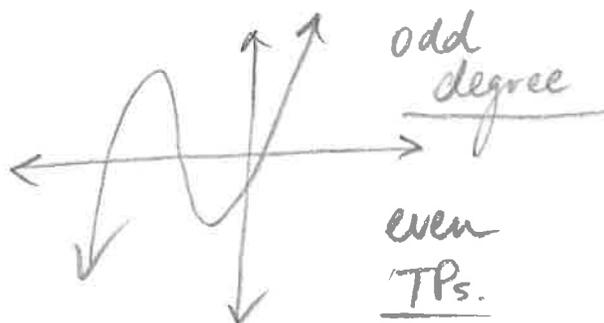
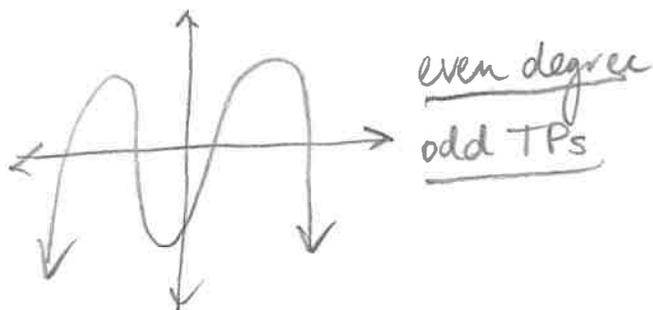
Turning Points of a Polynomial Function

- a polynomial function of degree n has, at most, $n-1$ turning points. (TPs)

Further info:

- an EVEN degree fxn has an ODD number of TPs.
- an ODD degree fxn has an EVEN number of TPs.

eg:



eg 5: For each of the following, provide the required info:

a) $y = -2x^5 + \dots$

b) $y = 3x^4 + \dots$

End Behaviour: UP, DOWN

UP, UP

Max. Real Roots: 5

4

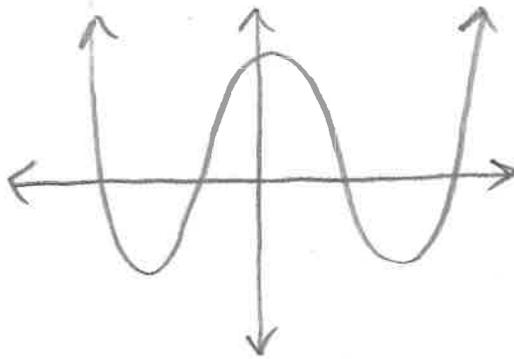
Min. Real Roots: 1

0

Max. Turning Pts.: 4

3

c)



Min. Degree: 4

*(b/c 3 TPs)

Multiplicity of Polynomial Roots

If a determined root is repeated r times, that root has a multiplicity of r .

Refer back to example 4 a and b:

a) each root has a MULT. of 1

b) 0 has a MULT. of 2.

eg 6: Given the following polynomial f(x)s,
find their roots and each root's
respective multiplicity:

a) $y = x^4 + x^3 - 6x^2$

$$0 = x^2(x^2 + x - 6)$$

$$0 = x^2(x+3)(x-2)$$

$$x = 0, 0, -3, 2$$

$x = 0, -3, 2$
②, ①, ①

b) $f(x) = x^5 + x^4 - 2x^3 - 2x^2 + x + 1$

$$0 = (x^5 + x^4) - 2x^3 - 2x^2 + x + 1$$

$$0 = x^4(x+1) - 2x^2(x+1) + 1(x+1)$$

$$0 = (x+1)(x^4 - 2x^2 + 1)$$

$$0 = (x+1)(x^2-1)(x^2-1)$$

$$0 = (x+1)(x+1)(x-1)(x+1)(x-1)$$

$$x = -1, -1, -1, 1, 1$$

$x = -1, 1$
③, ②

p. 119
#1-9

Ch. 3.2 - Graphing Polynomial Functions

Graphing roots of different multiplicities:

- i) Roots with a multiplicity of ONE:
 - the graph will CROSS the x -axis at those roots.
- ii) Roots with a multiplicity of TWO (or any even number):
 - the graph will BOUNCE off of the x -axis at those roots (ie. it touches, but does not CROSS).
- iii) Roots with a multiplicity of THREE (or odd # > 1):
 - the graph will CROSS the x -axis at those roots (however, it flattens out near the root (ie. it seems like it is going to bounce, but does not)).

Generally, an odd multiplicity has the graph cross the x -axis, whereas an even multiplicity has the graph bounce off of the x -axis.

STEPS TO GRAPHING A POLYNOMIAL FUNCTION:

- ① Determine the end behaviour by looking at the leading coefficient (is it \oplus or \ominus ?) and the degree of the function (use the degree to estimate # of turning points);
- ② Find the x -intercepts (set $f(x)/y = 0$ and solve);
- ③ Find the y -intercept (set $x = 0$ and solve);
- ④ Using the x -intercepts' multiplicities, determine the general shape at each root;

(5) Use the x -values around halfway between x -intercepts to estimate the relative maxima and minima ('highs and lows') of the graph;
* may be PRECISELY done with calculus.

(6) Draw a smooth, continuous curve connecting the points.

eg! Graph $f(x) = (x+1)(x-4)(x+3)$

END BEHAVIOUR: $f(x) = x^3 + \dots$

- An is POSITIVE } down on (L)
- DEGREE is ODD } up on (R)
- most likely, 2 turning points

x -ints: $0 = (x+1)(x-4)(x+3)$

$x = -1, 4, -3$ all w/ multiplicity of one.

y -int: $y = (0+1)(0-4)(0+3)$

$y = -12$ $(0, -12)$

PLOT points

Relative max/min: try $x = -2$
(estimate)

$y = (-2+1)(-2-4)(-2+3)$

$y = (-1)(-6)(1)$

$y = 6$ $(-2, 6)$

try $x = 2$

$y = (2+1)(2-4)(2+3)$

$y = (3)(-2)(5)$

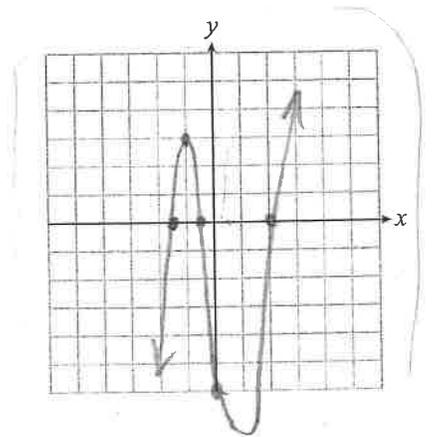
$y = -30$ yikes

try $x = 3$

$y = (3+1)(3-4)(3+3)$

$y = (4)(-1)(6)$

$y = -24$



eg2: Graph $y = (x+1)^2(1-x)(x-2)$

END BEHAVIOUR: $y = -x^4 + \dots$

- $a_n = -1$
 - DEGREE = 4 (even)
 - most likely, 3 turning points
- } down (L)
} down (R)

x-ints: $0 = (x+1)^2(1-x)(x-2)$

$$x = \underbrace{-1, -1}_{\text{TWO}}, \underbrace{1, 2}_{\text{ONE ea.}}$$

y-int: $y = (0+1)^2(1-0)(0-2)$

$$y = (1)(1)(-2)$$

$$y = -2 \quad (0, -2)$$

PLOT Points

Relative max/min:

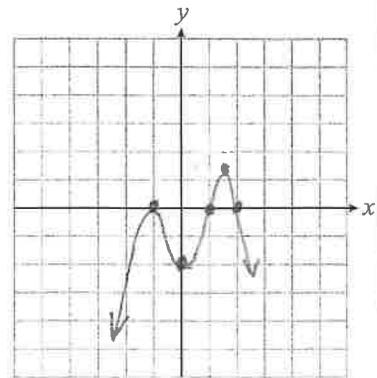
* y-int. is one of them $(0, -2)$

try $x = \frac{3}{2}$

$$y = \left(\frac{3}{2} + 1\right)^2 \left(1 - \frac{3}{2}\right) \left(\frac{3}{2} - 2\right)$$

$$y = \left(\frac{25}{4}\right) \left(-\frac{1}{2}\right) \left(-\frac{1}{2}\right)$$

$$y = \left(\frac{25}{4}\right) \left(\frac{1}{4}\right) = \frac{25}{16} \quad \left(\frac{3}{2}, \frac{25}{16}\right)$$



eg 3: Graph $y = -x(x+1)(x+2)^3$

END BEHAVIOUR: $y = -x^5 + \dots$

- $a_n = -1$
 - DEGREE = 5 (odd)
- } up on (L)
down on (R)
- most likely, 4 turning points

x-ints: $0 = -x(x-1)(x+2)^3$

$x = \underbrace{0, 1}_{\text{ONE ea.}} \quad \underbrace{-2, -2, -2}_{\text{THREE}}$

y-int: $y = -0(0-1)(0+2)^3$

$y = 0 \quad (0, 0)$

PLOT Points

Relative max/min:

try $x = -1$

$y = -(-1)(-1-1)(-1+2)^3$

$y = (1)(-2)(1)$

$y = -2 \quad (-1, -2)$

try $x = \frac{1}{2}$

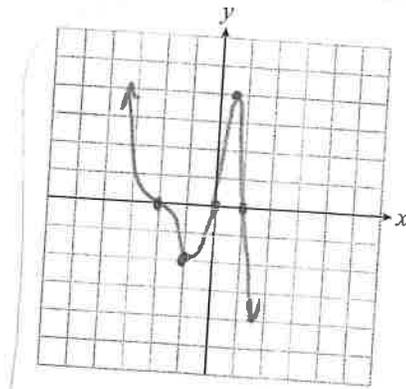
$y = -(\frac{1}{2})(\frac{1}{2}-1)(\frac{1}{2}+2)^3$

$y = (-\frac{1}{2})(-\frac{1}{2})(\frac{125}{8})$

$y = (\frac{1}{4})(\frac{125}{8})$

$y = \frac{125}{32} = \sim 4$

$(\frac{1}{2}, \sim 4)$



eg4: Graph $y = x^2(x^2+1)$

END BEHAVIOUR: $y = x^4 + \dots$

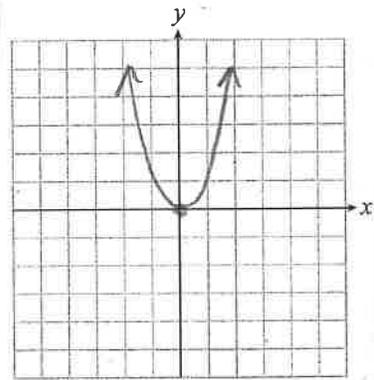
- $a_n = 1$
 - DEGREE = 4 (even)
 - most likely, 3 turning points.
- } up on (L)
} up on (R)

x-ints. : $0 = x^2(x^2+1)$

$x = \underbrace{0, 0}_{\text{TWO}} \quad (x^2+1) = 0$ has no solutions!

y-int: $y = 0^2(0^2+1)$
 $y = 0 \quad (0, 0)$

PLOT points



EQUATION OF POLYNOMIAL FUNCTIONS

eg5: A polynomial has roots of $-1, -1, 0,$ and 2 and $f(1) = 5$. What is its equation?

$$f(x) = a(x+1)(x+1)(x)(x-2)$$

$$f(x) = a(x+1)^2(x)(x-2)$$

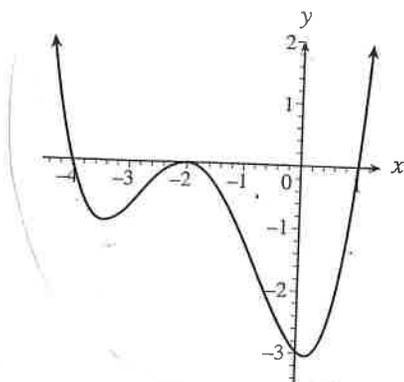
$$5 = a(1+1)^2(1)(1-2)$$

$$5 = -4a$$

$$a = -\frac{5}{4}$$

$$f(x) = -\frac{5}{4}x(x+1)^2(x-2)$$

eg6: Write an equation of a polynomial in lowest degree that represents the following graph:



x-intercepts:

-4 (ONE)

-2 (TWO)

1 (ONE)

y-intercept = $(0, -3)$ ie. when $x=0$, $y=-3$

$$f(0) = -3$$

$$f(x) = a(x+4)(x+2)^2(x-1)$$

$$-3 = a(0+4)(0+2)^2(0-1)$$

$$-3 = -16a$$

$$a = \frac{3}{16}$$

$$f(x) = \frac{3}{16}(x+2)^2(x+4)(x-1)$$

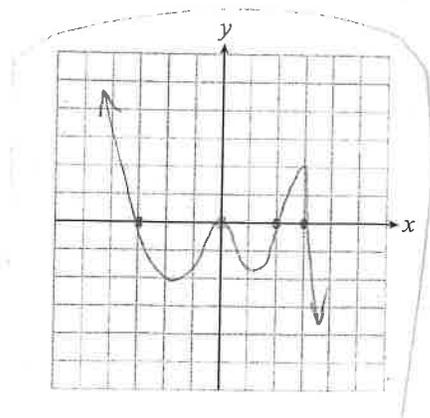
eg7: Given the following 'sign diagram', sketch the graph of a polynomial function in the lowest degree:

Value of $f(x)$	+	-	-	+	-
Zeros of x	-3	0	2	3	

0 has a multiplicity of Two

$$f(x) = a(x+3)(x^2)(x-2)(x-3)$$

also, up on (L) } $a < 0$
down on (R)



ESTIMATING ZEROS WITH/WITHOUT A GRAPHING CALCULATOR:

eg 8. Find a positive zero of $f(x) = x^5 + 2x^4 - 6x^3 + 2x - 3$ to one decimal place.

WITHOUT: $f(0) = -3$

$$f(1) = 1^5 + 2(1)^4 - 6(1)^3 + 2(1) - 3 = -4$$

$$f(2) = 2^5 + 2(2)^4 - 6(2)^3 + 2(2) - 3 = 17$$

} both neg.
(no cross)
x-intercept btwn 1 and 2

TRY $f(1.6) = -0.7834$

$$f(1.7) = 1.8248$$

zero btwn. 1.6 & 1.7

WITH: $(1.6341094, 0)$

* using "zero" function under "CALC"

p. 127 # 1-8

* for Q6, refer to example 7 p. 126

Ch. 3.3 - Division of Polynomials

DIVISION TERMINOLOGY:

$$\begin{array}{r} 3 \leftarrow \text{quotient} \\ 2 \overline{) 7} \leftarrow \text{dividend} \\ \underline{6} \\ 1 \leftarrow \text{remainder} \end{array}$$

$$2 \times 3 + 1 = 7$$

DIVISOR \times QUOTIENT + REMAINDER
= DIVIDEND

Generally:

$$\begin{array}{r} q(x) \\ x-a \overline{) p(x)} \\ \underline{\quad} \\ r \end{array}$$
$$p(x) = (x-a)(q(x)) + r$$

TWO TYPES OF DIVISION:

Long Division:

$$\begin{array}{r} 6x^2 - 7x - 3 \\ (x-2 \overline{) 6x^3 - 19x^2 + 11x + 6} \\ \ominus 6x^3 - 12x^2 \quad \downarrow \\ \hline -7x^2 + 11x \\ -7x + 14x \quad \downarrow \\ \hline -3x + 6 \\ -3x + 6 \\ \hline 0 \end{array}$$

Since $r = 0$, $x-2$ divides evenly into $6x^3 - 19x^2 + 11x + 6$.

Thus, $x-2$ is a factor of $6x^3 - 19x^2 + 11x + 6$.

This also means that $x=2$ is a zero of this polynomial because $P(2) = 0$.

$$\begin{aligned}
 \text{So, } 6x^3 - 19x^2 + 11x + 6 &= (x-2)(6x^2 - 7x - 3) \\
 &= (x-2)(6x^2 - 9x + 2x - 3) \\
 &= (x-2)[3x(2x-3) + 1(2x-3)] \\
 &= (x-2)(2x-3)(3x+1)
 \end{aligned}$$

$\uparrow \quad \uparrow \quad \uparrow$
 all are factors

Zeros : $x = 2, \frac{3}{2}, -\frac{1}{3}$

eg/:

Divide $3x^3 - 2x^2 + 1$ by $x - 2$

$$\begin{array}{r}
 x-2 \overline{) \begin{array}{r} 3x^3 - 2x^2 + 0x + 1 \\ 3x^3 - 6x^2 \\ \hline 4x^2 + 0x \\ 4x^2 - 8x \\ \hline 8x + 1 \\ 8x - 16 \\ \hline 17 \end{array} \\
 \end{array}$$

* $0x$ is a PLACEHOLDER

$$3x^3 - 2x^2 + 1 = (x-2)(3x^2 + 4x + 8) + 17$$

OR

$$\frac{3x^3 - 2x^2 + 1}{x-2} = 3x^2 + 4x + 8 + \frac{17}{x-2}$$

Synthetic Division:

- derived from LONG DIVISION! (see p.132)

eg 1: (again) $3x^3 - 2x^2 + 1 \div (x-2)$ using SYNTHETIC DIV.

Zero/root/x-int.
of the
DIVISOR

$$\begin{array}{r|rrrr} 2 & 3 & -2 & 0 & 1 \\ & \downarrow & 6 & 8 & 16 \\ \hline & 3 & 4 & 8 & 17 \\ & \downarrow & \downarrow & \downarrow & \downarrow \\ & \text{etc... } x^2 & \text{QUOTIENT } x & \text{Constant} & \text{REMAINDER} \end{array}$$

$$(x-2)(3x^2 + 4x + 8) + 17 = 3x^3 - 2x^2 + 1$$

eg 2: Divide $P(x) = 4x^5 - 30x^3 - 50x + 2$ by $x+3$

$$\begin{array}{r|rrrrrr} -3 & 4 & 0 & -30 & 0 & -50 & 2 \\ & \downarrow & -12 & 36 & -18 & 54 & -12 \\ \hline & 4 & -12 & 6 & -18 & 4 & -10 \\ & x^4 & x^3 & x^2 & x & x^0 & r \end{array}$$

$$P(x) = (x+3)(4x^4 - 12x^3 + 6x^2 - 18x + 4) - 10$$

OR

$$\frac{P(x)}{x+3} = 4x^4 - 12x^3 + 6x^2 - 18x + 4 - \frac{10}{x+3}$$

eg 3: Divide $P(x) = 6x^4 - 7x^3 + 4x^2 - 11x + 9$ by $2x - 1$.

*Note: $2x - 1 = 2(x - \frac{1}{2})$

$$\begin{array}{r|rrrrr} \frac{1}{2} & 6 & -7 & +4 & -11 & +9 \\ & & 3 & -2 & 1 & -5 \\ \hline & 6 & -4 & 2 & -10 & \boxed{4} \end{array}$$

So, $P(x) = (x - \frac{1}{2})(6x^3 - 4x^2 + 2x - 10) + 4$

BUT, we wanted to divide by $2x - 1$, so divide the quotient by 2 in order to double the divisor:

$$P(x) = (2x - 1)(3x^3 - 2x^2 + x - 5) + 4$$

$$\frac{P(x)}{2x - 1} = 3x^3 - 2x^2 + x - 5 + \frac{4}{2x - 1}$$

* so, when leading coefficient of divisor binomial is a (where $a \neq 1$), divide the coefficients of the quotient by a , after synthetically dividing.

ie. When synthetically dividing with a fraction, divide the quotient by the fraction's denominator.

eg4: Find k by synthetic division such that $2x^3 + x^2 - 5x + k \div (x+1)$ has a remainder of -3 .

$$\begin{array}{r|rrrr} -1 & 2 & 1 & -5 & k \\ & \downarrow & -2 & 1 & 4 \\ \hline & 2 & -1 & -4 & -3 \end{array} \quad \left. \vphantom{\begin{array}{r|rrrr} -1 & 2 & 1 & -5 & k \\ & \downarrow & -2 & 1 & 4 \\ \hline & 2 & -1 & -4 & -3 \end{array}} \right\} \begin{array}{l} k+4 = -3 \\ \boxed{k = -7} \end{array}$$

eg5: Divide $x^4 + 9x^3 - 5x^2 - 36x + 4$ by $x^2 - 4$

* divisor is QUADRATIC! Synthetic div. will only work w/ LINEAR binomials.

$x^2 - 4$ factors to $(x+2)(x-2)$

DIVIDE by both in succession:

$$\begin{array}{r|rrrrr} -2 & 1 & 9 & -5 & -36 & 4 \\ & & -2 & -14 & 38 & -4 \\ \hline & 1 & 7 & -19 & 2 & 0 \end{array}$$

$$\begin{array}{r|rrrrr} 2 & 1 & 7 & -19 & 2 & 0 \\ & & 2 & 18 & -2 & 0 \\ \hline & 1 & 9 & -1 & 0 & 0 \end{array}$$

(or, use LONG div.)

* if divisor is unable to be factored to binomials, you must use long division

$$\frac{P(x)}{x^2-4} = \boxed{x^2 + 9x - 1}$$

$$P(x) = (x^2 - 4)(x^2 + 9x - 1)$$

p. 135

1-4

$$\begin{array}{r}
 x^2 - 9 \quad \left[\begin{array}{l} 6x^2 - 5x + 56 \\ 6x^4 - 5x^3 + 2x^2 - 10x + 6 \\ 6x^4 - \cancel{54x^2} - 54x^2 \end{array} \right. \\
 \hline
 \cancel{29} \\
 -5x^3 + 56x^2 - 10x \\
 -5x^3 + 0x + 45x \\
 \hline
 56x^2 - 55x + 6 \\
 56x^2 + 0 - \cancel{504} \\
 \hline
 -55x + 510
 \end{array}$$

~~| | | | | | |
|----|---|-----|----|------|------|
| 3 | 6 | -5 | 2 | -10 | 6 |
| | | 18 | 39 | 223 | 639 |
| -3 | 6 | 13 | 41 | 213 | 645 |
| | | -18 | 15 | -162 | -153 |
| | 6 | -5 | 56 | 51 | |~~

3	6	-5	2	-10	6
		18	39	123	339
-3	6	13	41	113	345
		-18	15	-168	165
	6	-5	56	-55	510

Ch. 3.4 - The Remainder Theorem and The Factor Theorem

The Remainder Theorem

When dividing a polynomial by a binomial, the remainder can be determined using information provided by the binomial.

Say: $x-a \overline{) \begin{array}{l} q(x) \\ f(x) \\ \hline r \end{array}}$

so $f(x) = (x-a)(q(x)) + r$

Let $x = a$

$$f(a) = (a-a)(q(a)) + r$$

$$= 0(q(a)) + r$$

$$= 0 + r$$

$$\boxed{= r}$$

Thus, the value of the polynomial at $x = a$ is equal to the remainder of the polynomial when it is divided by $x - a$.

The Remainder Theorem: If the polynomial $P(x)$ is divided by $x - a$, the remainder is $P(a)$.

Corollary: If $P(x)$ is divided by $bx - a$, the remainder is $P\left(\frac{a}{b}\right)$.

eg1: Using the Remainder Theorem, find the remainder of each of the following divisions. Then, using YES or NO, answer the question, "Is the binomial a factor of the polynomial?"

a) $P(x) = 2x^4 - 3x^3 + 2x - 3 \div (x - 2)$

$a = 2$

$$P(2) = 2(2)^4 - 3(2)^3 + 2(2) - 3$$

$$= 2(16) - 3(8) + 4 - 3$$

$$= 9$$

NO since $r \neq 0$

b) $Q(x) = 2x^3 - 3x^2 - 23x + 12 \div (2x - 1)$

$$\frac{a}{b} = \frac{1}{2}$$

$$Q\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)^3 - 3\left(\frac{1}{2}\right)^2 - 23\left(\frac{1}{2}\right) + 12$$

$$= 2\left(\frac{1}{8}\right) - 3\left(\frac{1}{4}\right) - \frac{23}{2} + 12$$

$$= 0$$

YES since $r = 0$

c) $R(x) = x^{17} - 2x^{12} + 7 \div (x + 1)$

$$a = -1$$

$$R(-1) = (-1)^{17} - 2(-1)^{12} + 7$$

$$= -1 - 2 + 7$$

$$= 4$$

NO since $r \neq 0$

* can check w/ synthetic div. (tough w/ c)

eg2: For what value of k will the remainder equal 5 when $P(x) = x^3 - 2x^2 + x + k$ is divided by $x - 2$?

$$P(2) = 5 = (2)^3 - 2(2)^2 + 2 + k$$

$$5 = 8 - 8 + 2 + k$$

$$\boxed{k = 3}$$

Check!

The Factor Theorem

If a polynomial $P(x)$ is divided by $(x - a)$ and $r = 0$, then $x - a$ is a factor of $P(x)$ (ie. $x - a$ divides evenly into $P(x)$).

eg3: Is $x + 2$ a factor of $P(x) = 3x^4 + 4x^3 - 3x^2 - 3x - 10$?

* can use long div., synthetic div., or remainder thm.

$$\begin{array}{r|rrrrr} -2 & 3 & 4 & -3 & -3 & -10 \end{array}$$

$$\begin{array}{r|rrrrr} & & -6 & 4 & -2 & 10 \end{array}$$

$$\begin{array}{r|rrrrr} & 3 & -2 & 1 & -5 & \boxed{0} \end{array}$$

Yes!

$$P(x) = (x + 2)(3x^3 - 2x^2 + x - 5)$$

Note: -2 is a factor of -10

eg 4: Is $x-1$ a factor of $P(x) = 3x^4 + 4x^3 - 3x^2 - 3x - 10$?

$$\begin{array}{r|rrrrr} 1 & 3 & 4 & -3 & -3 & -10 \\ & & 3 & 7 & 4 & 1 \\ \hline & 3 & 7 & 4 & 1 & \boxed{-9} \end{array} \quad \boxed{\text{No!}}$$

$$P(x) = (x-1)(3x^3 + 7x^2 + 4x + 1) - 9$$

So...

The Factor Theorem:

Polynomial $P(x)$ has a factor $x-a$ if and only if $\underline{P(a) = 0}$;

ie. ① If $P(a) = 0$, then $x-a$ is a factor of $P(x)$.

② If $x-a$ is a factor of $P(x)$, then $P(a) = 0$.

Rational Root Theorem

If $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ is a polynomial function with INTEGER coefficients, then every rational root (zero) of $f(x)$ has the form

$$\frac{p}{q} \quad \left(\text{where } p \text{ is a factor of the constant, } a_0, \text{ and } q \text{ is a factor of the leading coefficient, } a_n. \right)$$

The Rational Root Theorem, therefore, may be utilized to solve polynomial equations (ie. find roots (x-ints.) of polynomial functions).

Possible Rational Zeros = $\frac{p}{q}$ = $\frac{\text{factors of constant}}{\text{factors of leading coefficient}}$

eg5: Find the zeros of $f(x) = x^3 - 9x^2 + 20x - 12$

Possible zeros: (TEST-values for synthetic division)

$$p = \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12 \quad \left. \vphantom{p} \right\} \frac{p}{q} = p \text{ (since } q = 1)$$

$$q = \pm 1$$

$$\begin{array}{r|rrrr} 1 & 1 & -9 & 20 & -12 \\ & & 1 & -8 & 12 \\ \hline & 1 & -8 & 12 & 0 \end{array}$$

1 is a root

$(x-1)$ is a factor

so is $x^2 - 8x + 12$ (quotient)

$$(x-6)(x-2)$$

$$f(x) = (x-1)(x-6)(x-2) = 0$$

$$\boxed{x = 6, 2, 1}$$

eg6: Find the x-intercepts when $f(x) = 4x^3 + 12x^2 + 5x - 6$ is graphed.

Possible zeros: $p = \pm 1, \pm 2, \pm 3, \pm 6$ } $\frac{p}{q}$ ^{Test-values} = $\pm 1, \pm 2, \pm 3, \pm 6, \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{3}{4}, \pm \frac{1}{4}$

$$q = \pm 1, \pm 2, \pm 4$$

$$\begin{array}{r|rrrr} -2 & 4 & 12 & 5 & -6 \\ & & -8 & -8 & 6 \\ \hline & 4 & 4 & -3 & 0 \end{array}$$

-2 is a root.

$x+2$ is a factor

So is $4x^2 + 4x - 3$

$$(2x-1)(2x+3)$$

$$f(x) = (x+2)(2x-1)(2x+3) = 0$$

$$\boxed{x = -2, \frac{1}{2}, -\frac{3}{2}}$$

P. 143
#1-5

and
next
page!
worksheet

Chapter 3.5 – Polynomial Applications (Word Problems)

Many real-life situations can be modeled by polynomial functions. Solutions to these functions may be rational, and can be solved by methods learned in this Unit. Other solutions are purely irrational, requiring the help of a graphing calculator. Any word problems that appear on the Unit Test will not require the use of a graphing calculator. Thus, while graphing calculator

questions will be assigned in the homework, they are not testable. *That said, any word problem 'theme' is testable, but would not require a graphing calc. to solve. (see eg. 4)*

Eg1: A box is constructed such that the length is twice the width and the height is 2 cm longer than the width. The volume of the box is 350 cm^3 . Find the dimensions of the box.

$$V(\text{box}) = lwh$$

Let $x = \text{width}$

then $2x = \text{length}$

and $x+2 = \text{height}$

$$V = x(2x)(x+2)$$

$$350 = 2x^3 + 4x^2$$

$$0 = 2x^3 + 4x^2 - 350$$

POSSIBLE ROOTS: MANY! BUT DO NOT CONSIDER NEGATIVES.

$$\begin{array}{r|rrrr} 5 & 2 & 4 & 0 & -350 \\ & & 10 & 70 & 350 \\ \hline & 2 & 14 & 70 & 0 \end{array}$$

5 is a root

$(x-5)$ is a factor

so is $2x^2 + 14x + 70$

solve $2x^2 + 14x + 70 = 0$

$$x = \frac{-14 \pm \sqrt{14^2 - 4(2)(70)}}{4}$$

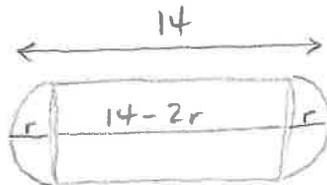
$$= \frac{-14 \pm \sqrt{-364}}{4}$$

No solution.

Only root: $x = 5$

WIDTH = 5 cm
 LENGTH = 10 cm
 HEIGHT = 7 cm

Eg2: A vitamin capsule has the shape of a right circular cylinder with hemispheres on each end. The total length of the capsule is 14 mm and its volume is $108\pi \text{ mm}^3$. Find the radius of the capsule.



$$V(\text{cylinder}) = \pi r^2 h \quad V(\text{sphere}) = \frac{4}{3} \pi r^3$$

$$h = 14 - 2r$$

$$V = \pi r^2 h + \frac{4}{3} \pi r^3$$

two halves of a sphere make a whole sphere

$$\frac{108\pi}{\pi} = \frac{\pi r^2 (14 - 2r) + \frac{4}{3} \pi r^3}{\pi}$$

$$108 = r^2 (14 - 2r) + \frac{4}{3} r^3$$

$$0 = -2r^3 + 14r^2 + \frac{4}{3} r^3 - 108$$

$$0 = -\frac{2}{3} r^3 + 14r^2 - 108$$

$$0 = -2r^3 + 42r^2 - 324$$

$$0 = r^3 - 21r^2 + 162$$

$$\begin{array}{r|rrrr} 3 & 1 & -21 & 0 & 162 \\ & & 3 & -54 & -162 \\ \hline & 1 & -18 & -54 & 0 \end{array}$$

3 is a root

$x-3$ is a factor

→ Solve:

$$r^2 - 18r - 54 = 0$$

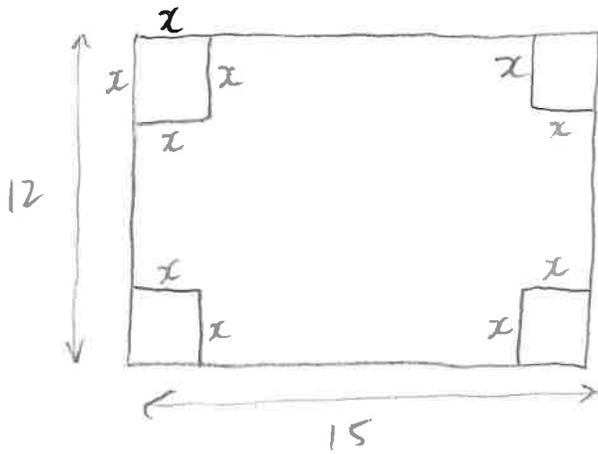
$$r = \frac{18 \pm \sqrt{(-18)^2 - 4(-54)}}{2}$$

$$r = 20.6, -2.6$$

r cannot be 20.6
($14 - 2r$) would be neg.

$\text{RADIUS}(r) = 3 \text{ mm}$

Eg3: An open rectangular box is constructed by cutting a square of length x from each corner of a 12 cm by 15 cm rectangular piece of cardboard, then folding up the sides. What is the length of the square that must be cut from each corner if the volume of the box is 112 cm^3 ? Note: x must be larger than 1.



$$V = lwh$$

$$h = x$$

$$l = 15 - 2x$$

$$w = 12 - 2x$$

$$112 = (15 - 2x)(12 - 2x)(x)$$

$$112 = 4x^3 - 54x^2 + 180x$$

$$0 = 4x^3 - 54x^2 + 180x - 112$$

$$0 = 2x^3 - 27x^2 + 90x - 56$$

$$\begin{array}{r|rrrrr} 4 & 2 & -27 & 90 & -56 & \\ & & 8 & -76 & 56 & \\ \hline & 2 & -19 & 14 & 0 & \end{array}$$

4 is a root
 $x - 4$ is a factor

Solve: $2x^2 - 19x + 14 = 0$

$$x = \frac{19 \pm \sqrt{(-19)^2 - 4(2)(14)}}{4} = 0.81, 8.69$$

$$0.81 < 1 \text{ (reject)}$$

$$15 - 2(8.69) < 0 \text{ (reject)}$$

$$h = x = 4 \text{ cm}$$

Eg4: The production of x Tesla Model S cars produces revenue of:

$$R(x) = 4x^2 + 6x$$

and costs:

$$C(x) = \frac{1}{4}x^3 - \frac{3}{4}x^2 - 4x + 100$$

At what point does Tesla make a profit? (ie. within what domain is a profit produced?)

Hint: Profit is gained when Revenue > Costs

$$R(x) > C(x)$$

$$4x^2 + 6x > \frac{1}{4}x^3 - \frac{3}{4}x^2 - 4x + 100$$

$$-\frac{1}{4}x^3 + \frac{19}{4}x^2 + 10x - 100 > 0$$

$$-x^3 + 19x^2 + 40x - 400 > 0$$

UP, DOWN

4	-1	19	40	-400
	↓	-4	60	400
	-1	15	100	0

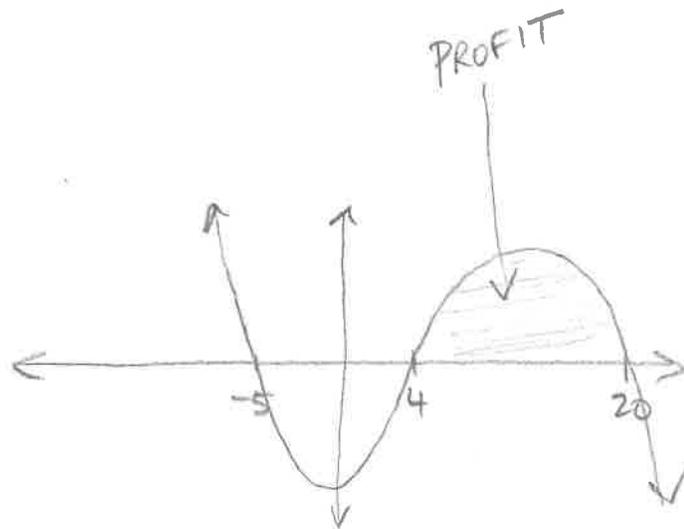
$$(x-4)(-x^2 + 15x + 100) = 0$$

$$-1(x-4)(x^2 - 15x - 100) = 0$$

$$(x-4)(x-20)(x+5) = 0$$

$$x = 4, 20, -5$$

$$y\text{-int. } (0, -100)$$



Profit is made

when $x > 4$, but
 $x < 20$

$$4 < x < 20$$