

Ch. 5.1 - Exponents and Exponential Functions

Review: Gr. 9 and 10 Properties:

let a and b be positive real numbers

let $x, y \in \mathbb{R}$

$$1. b^0 = \underline{1} \quad 2. b^x \cdot b^y = \underline{b^{x+y}}$$

$$3. \frac{b^x}{b^y} = \underline{b^{x-y}} \quad 4. (b^x)^y = \underline{b^{xy}}$$

$$5. \left(\frac{a}{b}\right)^{-x} = \underline{\left(\frac{b}{a}\right)^x} = \underline{\frac{b^x}{a^x}}$$

$$6. b^{-x} = \underline{\left(\frac{1}{b}\right)^x} = \underline{\frac{1^x}{b^x}} = \underline{\frac{1}{b^x}}$$

$$7. (ab)^x = \underline{a^x b^x}$$

$$8. a^x = a^y \text{ if and only if } \underline{x = y}.$$

note: $a \neq 0, 1$.

* note: We need to be aware of exponential 'factors' and 'multiples'!

eg: $2^0 = 1, 2^1 = 2, 2^2 = 4, 2^3 = 8 \dots$

1, 2, 4, 8, 16, 32, etc. are exponential 'multiples' of 2.

also: 2 is an exponential 'factor' of 1, 2, 4, 8, 16, etc.

eg1: What are the exponential 'factors' of 2 that are less than 1?

$$2^0 = 1, \text{ so } 2^{-1} = \frac{1}{2}, 2^{-2} = \frac{1}{4}, 2^{-3} = \frac{1}{8}$$

i.e. the reciprocals of 1, 2, 4, 8, etc.

eg2: Simplify $\frac{4^{6x+1}}{8^{4x+2}}$

$$\begin{aligned} &= \frac{(2^2)^{6x+1}}{(2^3)^{4x+2}} = \frac{2^{12x+2}}{2^{12x+6}} = 2^{2-6} \\ &= 2^{-4} \\ &= \boxed{\frac{1}{16}} \end{aligned}$$

eg3: Solve for x :

$$9^{2x-3} = 27^{1-x}$$

$$(3^2)^{2x-3} = (3^3)^{1-x}$$

$$3^{4x-6} = 3^{3-3x}$$

$$4x-6 = 3-3x$$

$$7x = 9$$

$$x = \frac{9}{7}$$

Graphing Exponential Functions

The basic general function: $y = b^x$,

ie. $y = b^{1(x-0)} + 0$

can be transformed to:

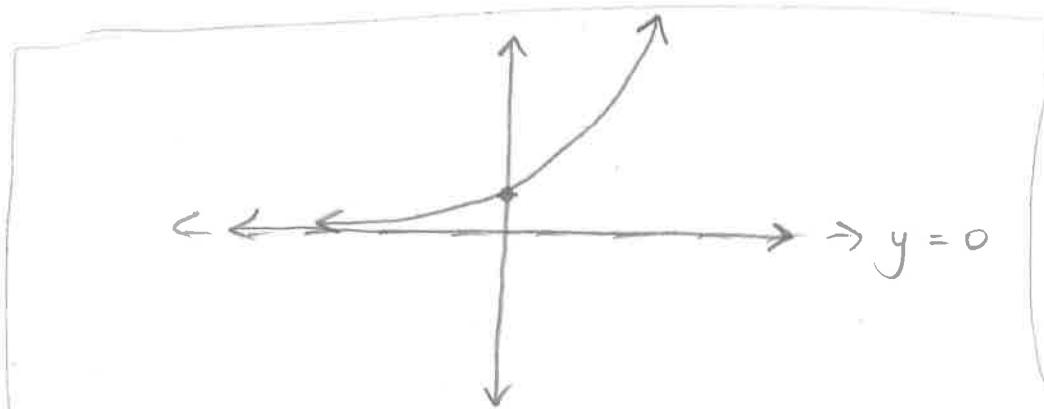
$y = ab^{n(x-c)} + d$

note: $b > 0$ and $b \neq 1$.

Exponential functions possess a horizontal asymptote of $y = d$.

Two Scenarios:

① When $b > 1$, $y = b^x$ looks like:

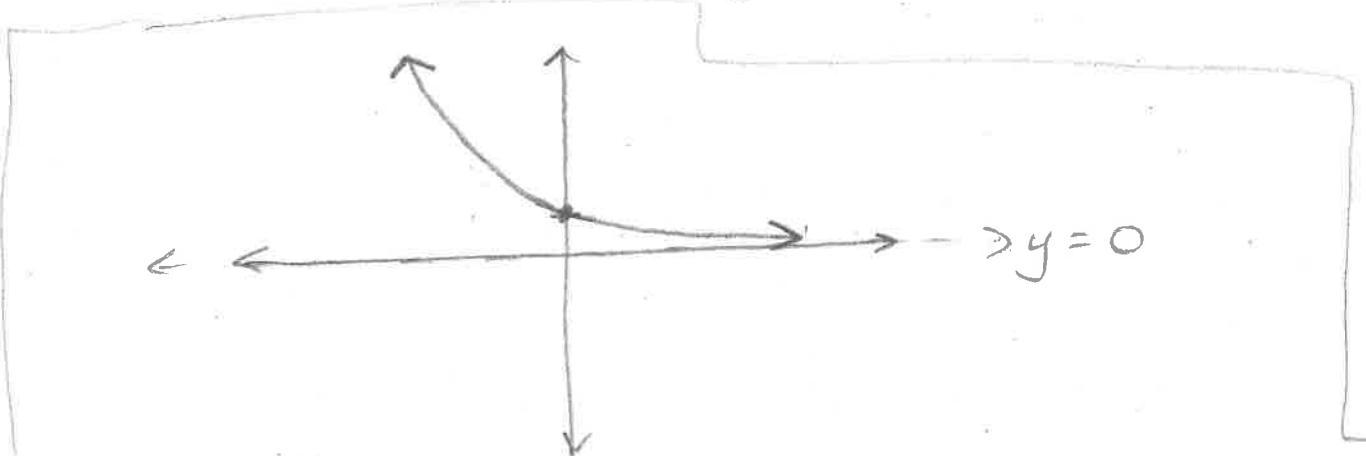


- it possesses the point $(0, 1)$

- it is INCREASING (from L to R)

} could change
with
transformations

② When $0 < b < 1$, $y = b^x$ looks like:



- it possesses the point $(0, 1)$
 - it is DECREASING
- } could change
with
transformations

eg4: Sketch each graph:

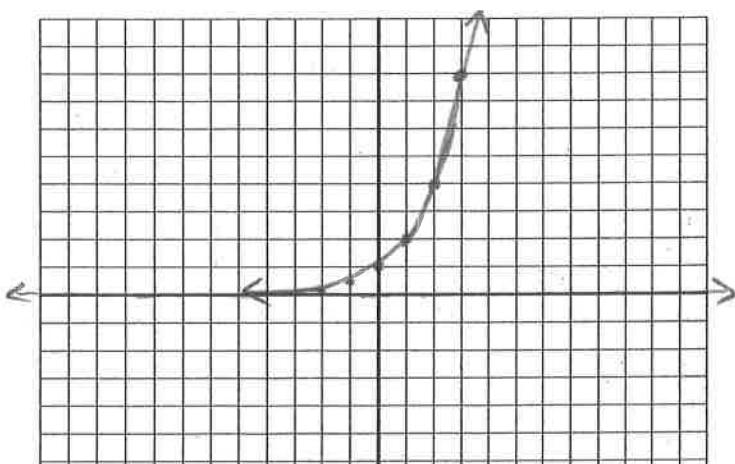
a) $y = 2^x$

$b = 2 > 1 \rightarrow$ increasing

i.e. the graph of $y = 2^x$ not transformed at all.

horizontal asymptote: $y = 0$

x	y
0	1
1	2
2	4
-1	$\frac{1}{2}$
-2	$\frac{1}{4}$
3	8



b) $y = -2^{x-3} + 1$

* $y = 2^x$

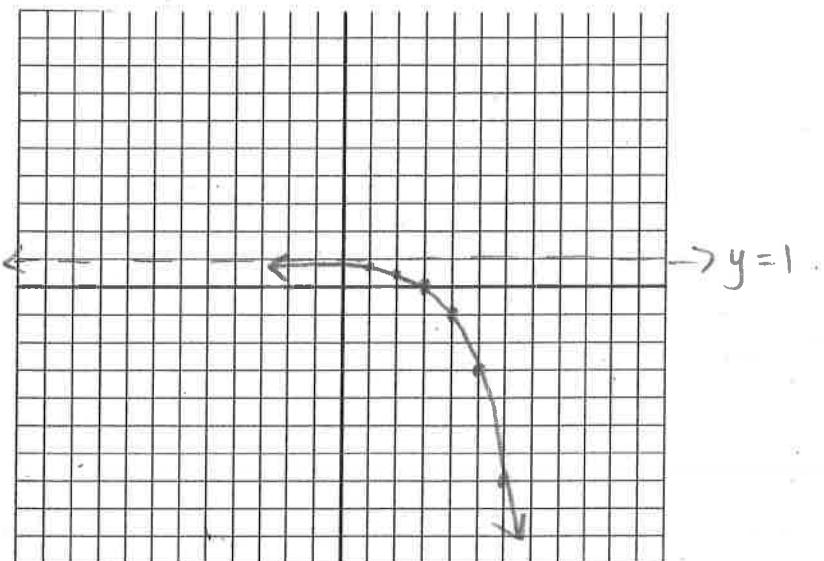
- reflected over x-axis

- 3 (R)

- 1 UP

horizontal asymptote: $y = 1$

x	y
$3+0$	$1 \times -1 + 1$
$3+1$	$2 \times -1 + 1$
$3+2$	$4 \times -1 + 1$
$3+3$	$8 \times -1 + 1$
$3+-1$	$\frac{1}{2} \times -1 + 1$
$3+-2$	$\frac{1}{4} \times -1 + 1$



c) $y = \left(\frac{1}{2}\right)^x$

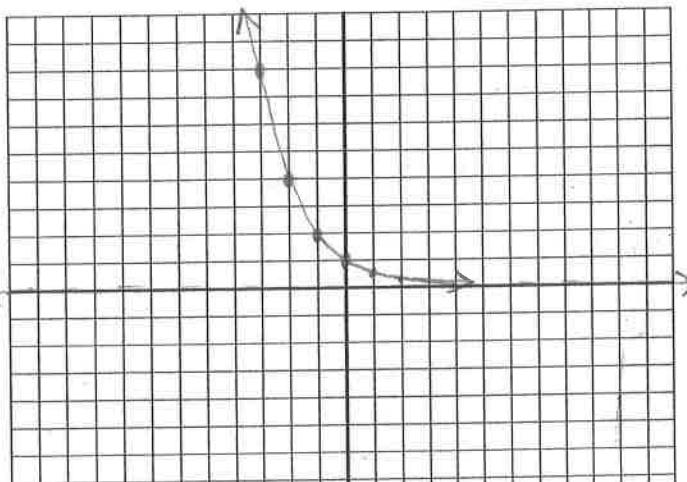
$b = \frac{1}{2} < 1$

\hookrightarrow decreasing

i.e. the graph of $y = \left(\frac{1}{2}\right)^x$ not transformed at all.

horizontal asymptote: $y = 0$.

x	y
0	1
1	$\frac{1}{2}$
2	$\frac{1}{4}$
-1	2
-2	4
-3	8



$$d) y = \left(\frac{1}{2}\right)^{x+2} - 3$$

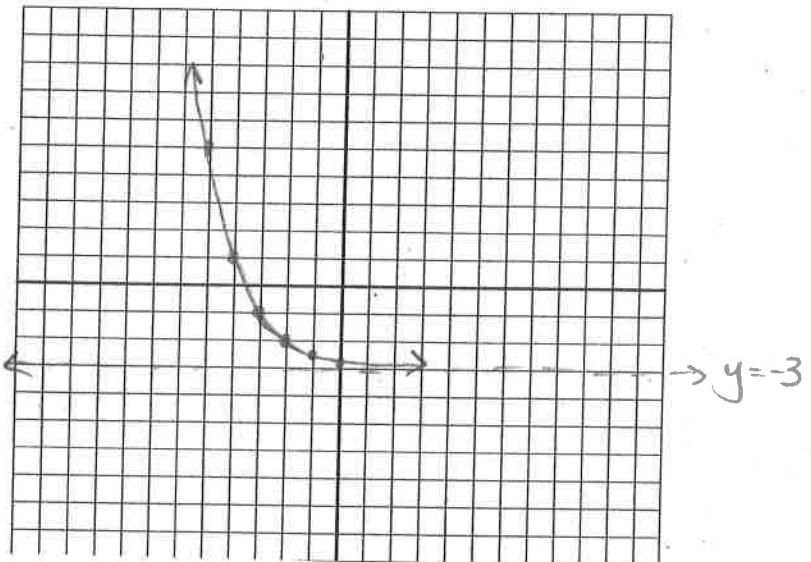
$$\times \quad y = \left(\frac{1}{2}\right)^x$$

- 2 ↙

- 3 DOWN

horizontal asymptote: $y = -3$

x	y
0^{-2}	$1 - 3$
1^{-2}	$\frac{1}{2} - 3$
2^{-2}	$\frac{1}{4} - 3$
-1^{-2}	$2 - 3$
-2^{-2}	$4 - 3$
-3^{-2}	$8 - 3$



Applications of Exponential Functions

- radioactive decay (including half-lives), bacterial growth, spread of disease (epidemics), compound interest, depreciation, etc...

COMPOUND INTEREST FORMULA

Compound Interest - interest calculated upon principal plus previously earned interest.

$$A = P \left(1 + \frac{r}{n}\right)^{nt}$$

A = Final amount

t = time in years

P = Principal (initial amount)

$$I = A - P$$

r = annual interest rate
(as a decimal)

I = interest earned

n = number of times the yearly
interest is compounded per year.

eg5: Find the interest earned if \$9000 is deposited into an account paying 2.5% interest compounded monthly for 10 years.

$$A = P \left(1 + \frac{r}{n}\right)^{nt} \quad I = A - P$$

$$A = 9000 \left(1 + \frac{0.025}{12}\right)^{(12)(10)} = 11553.22 \\ - 9000$$

$$A = 11553.22$$

$$= \$2553.22$$

eg6: What initial investment is required in order to become a millionaire in 30 years if you earn 6% interest compounded quarterly?

$$A = P \left(1 + \frac{r}{n}\right)^{nt}$$

$$P = \frac{A}{\left(1 + \frac{r}{n}\right)^{nt}}$$

$$P = \frac{1000000}{\left(1 + \frac{0.06}{4}\right)^{4(30)}}$$

$$P = \$167523.19$$

Growth and Decay Formula

$$A = A_0 (x)^{\frac{t}{T}}$$

A = Final amount

A_0 = Original amount

x = growth or decay value

eg: half-life $\rightarrow \frac{x=0.5}{\text{increases by } 10\% \rightarrow \frac{x=1.10}{\text{decreases by } 25\% \rightarrow \frac{x=0.75}}}$

T = the time required by A_0 to increase/decrease by the factor of x

t = total time that the item has grown/existed.

* make sure T and t are in the same units!

eg 7: The number of fruit flies increases by 40% every 3 days. How many will exist in 30 days if you begin with 10 flies?

$$A = A_0 (x)^{\frac{t}{T}}$$

$$A = 10 (1.40)^{\frac{30}{3}}$$

$$A = \boxed{289 \text{ flies}}$$

eg 8: The half-life of radioactive Plutonium-239 is about 25000 years. What percentage of a given sample will remain after 5000 years?

$$A = A_0 (x)^{\frac{t}{T}}$$

$$A = A_0 (0.5)^{\frac{5000}{25000}}$$

$$A = A_0 (0.871)$$

$$\frac{A}{A_0} = 0.871$$

87.1% remains

(12.9% decayed)

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Ch. 5.2 - Logarithmic Functions

- a logarithmic function is the inverse of a specific exponential function.

recall: with respect to Inverse functions (Ch. 2.5):

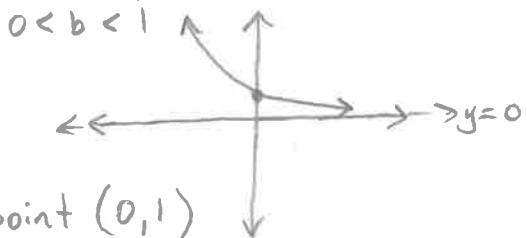
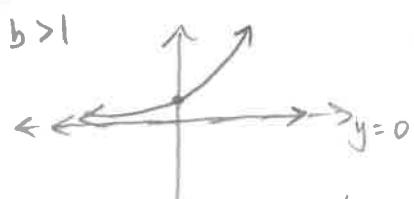
1. The original function must be one-to-one in order for the inverse to be a function.
2. $f^{-1}(x)$ interchanges the x and y -coordinates of the original $f(x)$.
3. The domain of $f(x)$ becomes the range of $f^{-1}(x)$.
4. The range of $f(x)$ becomes the domain of $f^{-1}(x)$.
5. The graphs of $f(x)$ and $f^{-1}(x)$ are a reflection over the line $y = x$.

Take the exponential function $y = b^x$:

its domain is $x \in \mathbb{R}$

its range is $y > 0$

its horizontal asymptote is $y = 0$



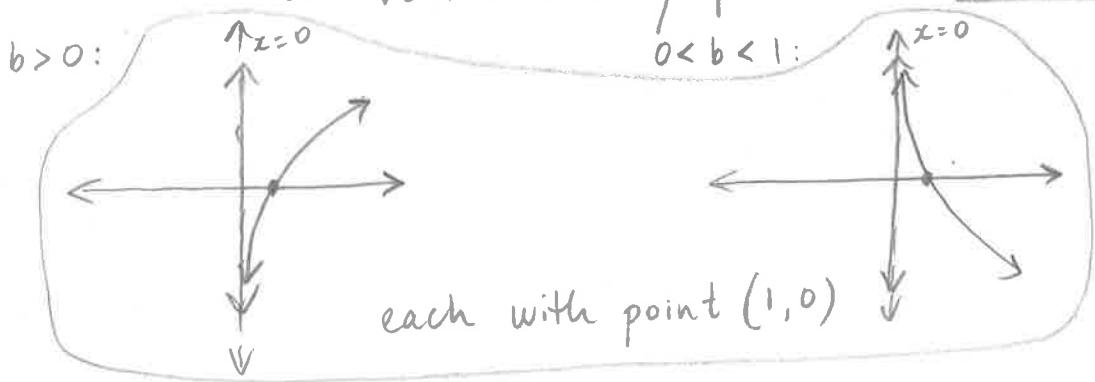
each with point $(0,1)$

So, the inverse function will have:

a domain of $x > 0$

a range of $y \in \mathbb{R}$

a vertical asymptote of $x = 0$



let's give these graphs a name!

$$f: f(x) = b^x \quad (b > 0, b \neq 1)$$

$$\underline{y = b^x}$$

$$f^{-1}: \begin{aligned} x &= b^y \\ \text{SOLVE FOR } y! & \\ y &= \log_b x \\ (b > 0, b \neq 1) & \\ f^{-1}(x) &= \log_b x \end{aligned}$$

we could use this to get
the above graphs, but we would
have to pick y first when
using a table of values.

Again:

$$y = b^x$$

inverse

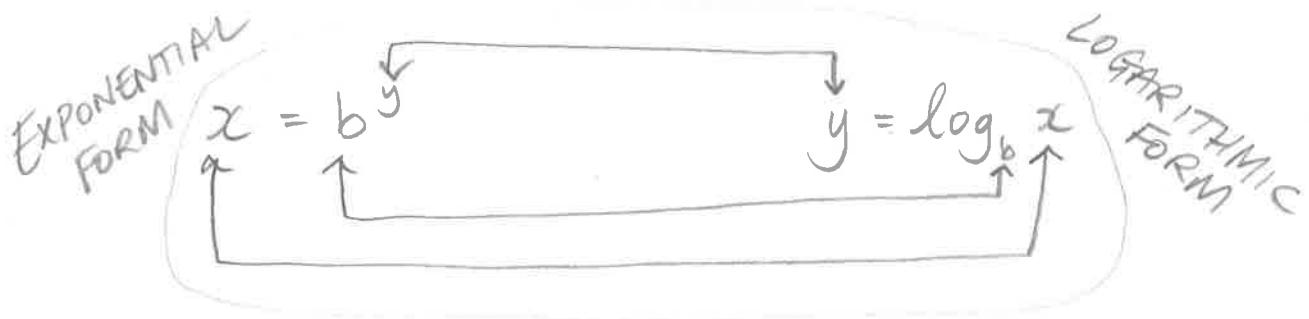
inverse

$$x = b^y$$

equivalent
statements

$$y = \log_b x$$

pick x first
when using
table of
values.



Therefore: a logarithm is important in order to solve for an unknown exponent.

NOTE: $\log_{10} x = \frac{\log x}{\log 10}$ } all bases other than 10 and e must be written as a subscript.

eg1: Change the following logarithmic equations to exponential equations:

a) $\log_4 2 = \frac{1}{2}$

$$4^{\frac{1}{2}} = 2$$

(ie. $\sqrt{4} = 2$)

b) $\log_2 \frac{1}{8} = -3$

$$2^{-3} = \frac{1}{8}$$

(ie. $\frac{1}{2^3} = \frac{1}{8}$)

eg2: Change the following exponential equations to logarithmic equations:

a) $3^4 = 81$

$$\log_3 81 = 4$$

b) $6^{-2} = \frac{1}{36}$

$$\log_6 \frac{1}{36} = -2$$

Ex 3: Solve each of the following for y :

a) $y = \log_4 32$

$$\begin{aligned} 4^y &= 32 \\ (2^2)^y &= 2^5 \\ 2y &= 5 \\ y &= \frac{5}{2} \end{aligned}$$

b) $y = \log_9 27$

$$\begin{aligned} 9^y &= 27 \\ (3^2)^y &= 3^3 \\ 2y &= 3 \\ y &= \frac{3}{2} \end{aligned}$$

Graphing Logarithmic Functions

The basic general function: $y = \log_b x$

i.e. $y = \log_b (1(x-0)) + 0$

can be transformed to:

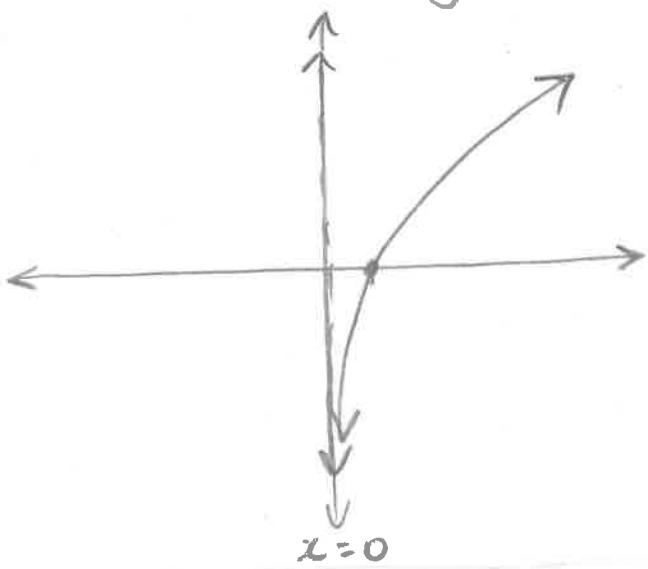
$y = a \log_b (n(x-c)) + d$

note: $b > 0$ and $b \neq 1$

Logarithmic functions possess a vertical asymptote of $x = c$.

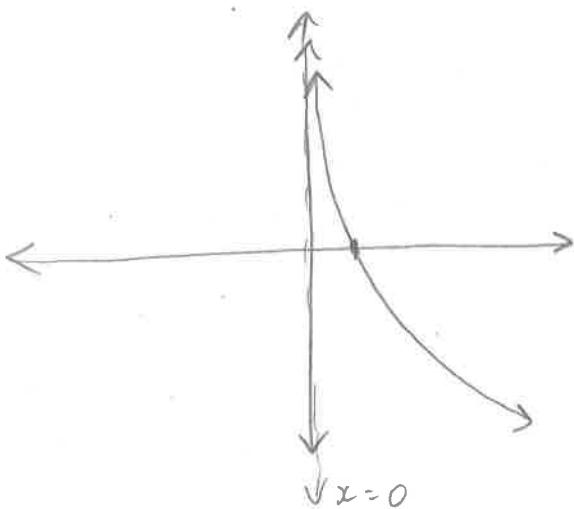
Two Scenarios:

① When $b > 1$, $y = \log_b x$ looks like:



- it possesses the point $(1, 0)$
 - it is INCREASING (from L to R)
- } could change with transformations

② When $0 < b < 1$, $y = \log_b x$ looks like:



- it possesses the point $(1, 0)$
 - it is DECREASING
- } could change with transformations

Note: each of graphs ① and ② can be deduced in either of the following ways:

- i) Graph $y = \log_b x$ by picking x first when using a table of values.
- ii) Graph the equivalent $x = b^y$ by picking y first when using a table of values.

eg 4: Sketch each graph:

a) $y = \log_2 x$

- Options:
- i) Graph the INVERSE, then swap coordinates
 - ii) Convert to equivalent exponential equation, then pick y first.
 - iii) Graph it as it reads by picking x first.

Note: this is the graph of $y = \log_2 x$ not transformed at all.

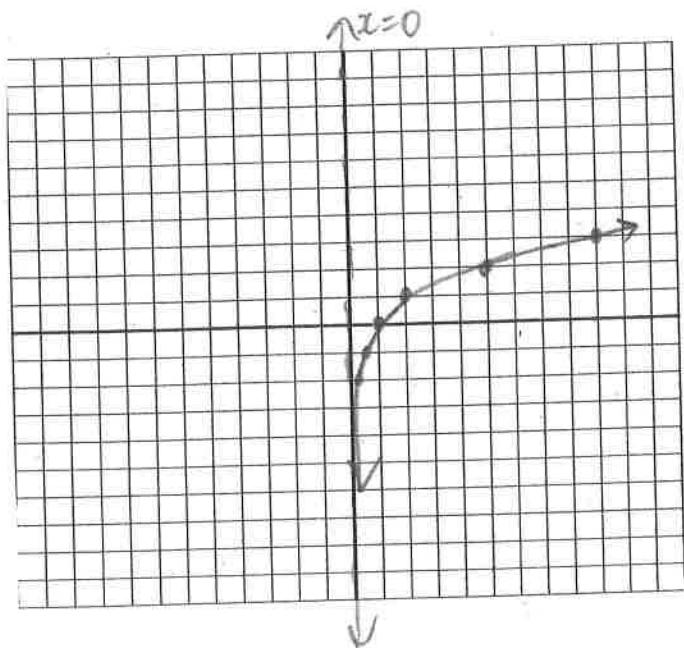
$b = 2 > 1 \rightarrow$ increasing

x	y
1	0
2	1
4	2
$\frac{1}{2}$	-1
$\frac{1}{4}$	-2

and...
ii) use $x = 2^y$ and pick y first!

* graph next page.

$$y = \log_2 x$$



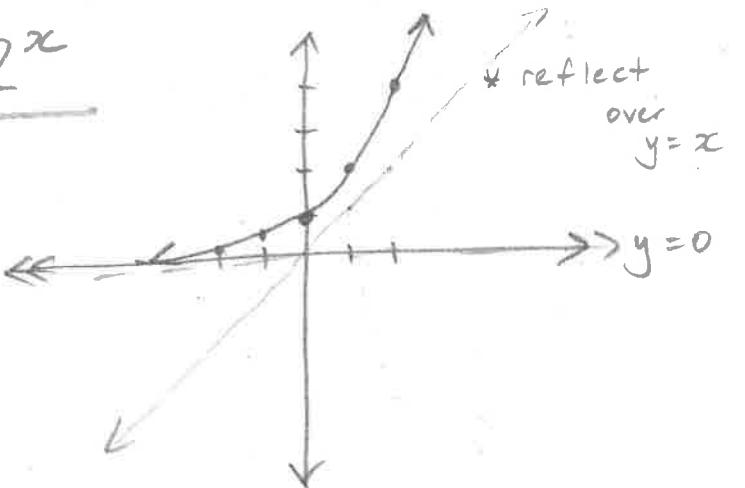
deductions?

x	y
1	0
2	1
4	2
$\frac{1}{2}$	-1
$\frac{1}{4}$	-2
8	3

$\log_2 1 = 0$
$\log_2 2 = 1$
$\log_2 4 = 2$
$\log_2 \frac{1}{2} = -1$
$\log_2 \frac{1}{4} = -2$
$\log_2 8 = 3$

Also, the graph above is the

inverse of $y = 2^x$



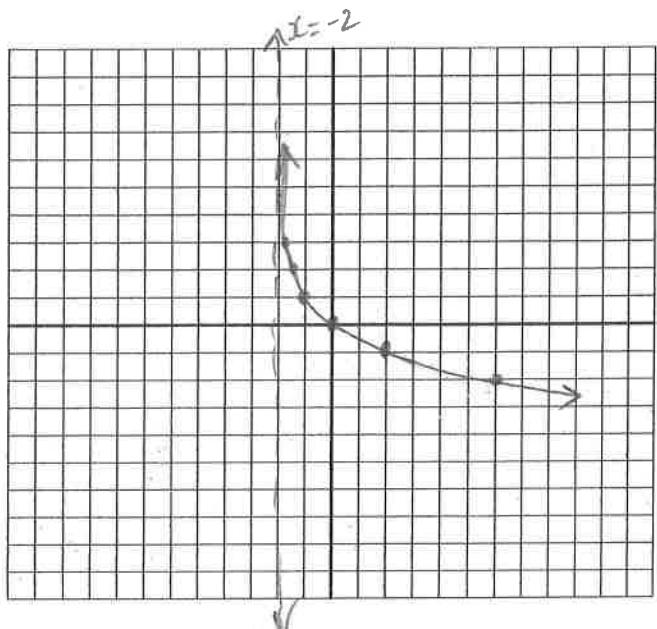
b) $y = -\log_2(x+2) + 1$

x	y
1^{-2}	$0 \times -1 + 1$
2^{-2}	$1 \times -1 + 1$
4^{-2}	$2 \times -1 + 1$
$\frac{1}{2}^{-2}$	$-1 \times -1 + 1$
$\frac{1}{4}^{-2}$	$-2 \times -1 + 1$
8^{-2}	$3 \times -1 + 1$

* $y = \log_2 x$

- reflected over x-axis
- 2 (L)
- 1 UP

vertical 'hole' @ $x = -2$



c) $y = \log_{\frac{1}{2}} x$

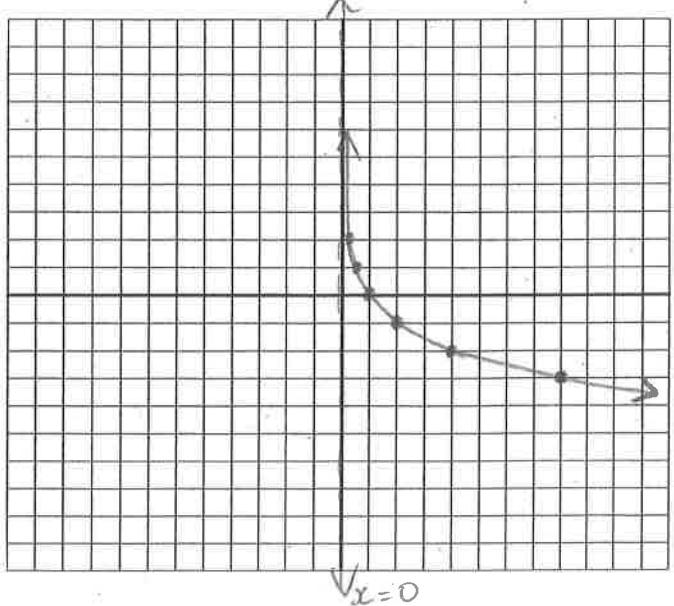
$b = \frac{1}{2} < 1 \rightarrow \text{decreasing}$

ie. the graph of $y = \log_{\frac{1}{2}} x$
not transformed at all.

- Vertical asymptote @ $x = 0$

$x = \left(\frac{1}{2}\right)^y$ pick y first

x	y
1	0
$\frac{1}{2}$	1
$\frac{1}{4}$	2
2	-1
4	-2
8	-3



d) $y = \log_{\frac{1}{2}}(x-1) - 4$

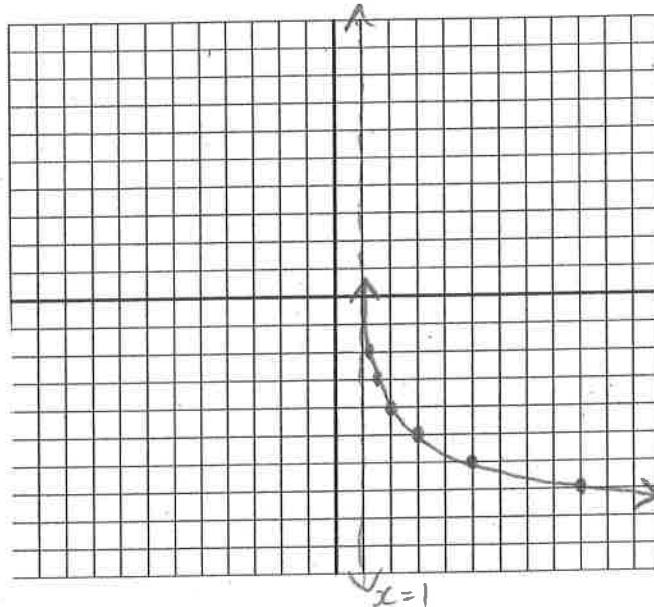
x	y
$1 + 1$	$0 - 4$
$1 + \frac{1}{2}$	$1 - 4$
$1 + \frac{1}{4}$	$2 - 4$
$1 + 2$	$-1 - 4$
$1 + 4$	$-2 - 4$
$1 + 8$	$-3 - 4$

* $y = \log_{\frac{1}{2}} x$

- 1 (R)

- 4 down

vertical 'tote' @ $x = 1$



eg5: What is the domain of $y = \log_{(x-1)}(x+2)$?

Using template of $y = \log_b x$

$$b > 0, b \neq 1, x > 0$$

$$x-1 > 0 \quad x-1 \neq 1 \quad x+2 > 0$$

$$x > 1 \quad x \neq 2 \quad x > -2$$

Therefore: DOMAIN: $x > 1, x \neq 2$

eg 6: For each of the following:

1. Find the inverse

2. Find domain and range of BOTH functions

a) $f(x) = 2^{x-1} + 3$

$$y = 2^{x-1} + 3$$

$$f^{-1}: x = 2^{y-1} + 3$$

$$x-3 = 2^{y-1}$$

$$y-1 = \log_2(x-3)$$

$$y = \log_2(x-3) + 1$$

$$\boxed{f^{-1}(x) = \log_2(x-3) + 1}$$

$$\begin{aligned} f: D: & x \in \mathbb{R} \\ R: & y > 3 \end{aligned}$$

$$\begin{aligned} f^{-1}: D: & x > 3 \\ R: & y \in \mathbb{R} \end{aligned}$$

b) $f(x) = -\log_5(x+1) - 3$

$$y = -\log_5(x+1) - 3$$

$$f^{-1}: x = -\log_5(y+1) - 3$$

$$x+3 = -\log_5(y+1)$$

$$-(x+3) = \log_5(y+1)$$

$$y+1 = 5^{-(x+3)}$$

$$y = 5^{-(x+3)} - 1$$

$$\boxed{f^{-1}(x) = 5^{-(x+3)} - 1}$$

$$f: D: x > -1$$

$$R: y \in \mathbb{R}$$

$$f^{-1}: D: x \in \mathbb{R}$$

$$R: y > -1$$

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Ch. 5.3 - Properties of Logarithms

Two Simple Properties:

i) Since $b^0 = 1$, $\underline{\log_b 1 = 0}$.

ii) Since $b^1 = b$, $\underline{\log_b b = 1}$.

Product Rule

Let $x = \log_b A$

and let $y = \log_b B$

then... $b^x = A$

and $b^y = B$

$$\text{so, } AB = b^x b^y = b^{x+y}$$

$$AB = b^{x+y}$$

$$\therefore \underline{\log_b AB = x + y}$$

$$\boxed{\log_b AB = \log_b A + \log_b B}$$

* can be utilized
in either direction

eg1: Simplify $\log 3x$

$$\log 3 \cdot x = \boxed{\log 3 + \log x}$$

eg2: Simplify $\log 4 + \log 6$

$$\log 4 + \log 6 = \log(4 \cdot 6) = \boxed{\log 24}$$

Quotient Rule

let $x = \log_b A$
and let $y = \log_b B$

then... $b^x = A$

and $b^y = B$

$$\frac{A}{B} = \frac{b^x}{b^y} = b^{x-y}$$

$$\frac{A}{B} = b^{x-y}$$

$$\log_b\left(\frac{A}{B}\right) = x - y$$

$$\boxed{\log_b\left(\frac{A}{B}\right) = \log_b A - \log_b B}$$

* can be utilized in
either direction

eg3: Simplify $\log\left(\frac{x}{2}\right)$

$$\log\left(\frac{x}{2}\right) = \boxed{\log x - \log 2}$$

eg4: Simplify $\log 12 - \log 4$

$$\log 12 - \log 4 = \log\left(\frac{12}{4}\right) = \boxed{\log 3}$$

Power Rule

$$\text{Let } x = \log_b A$$

$$\text{then } A = b^x$$

$$A^n = b^{nx}$$

$$\log_b A^n = nx$$

$$\boxed{\log_b A^n = n \log_b A}$$

q5: Simplify $\log_2 8$ using the Power Rule
(Evaluate)

$$\log_2 8 = \log_2 (2^3) = 3 \log_2 2$$

$$= 3(1) = \boxed{3}$$

Change of Base Rule

→ usually change to base 10

$$\text{Let } y = \log_b A$$

$$\text{then } A = b^y$$

$$\log_x A = \log_x b^y$$

$$\log_x A = y \log_x b \quad (\text{Power Rule})$$

$$y = \frac{\log_x A}{\log_x b}$$

$$\boxed{\log_b A = \frac{\log_x A}{\log_x b}}$$

eg 6: Find $\log_2 7$ to 3 decimal places.

Note: Calculators have a 'log' function, which means \log_{10} . They cannot calculate logs of bases other than 10. (one exception \Rightarrow later)

$$\log_2 7 = \frac{\log_{10} 7}{\log_{10} 2} = \frac{0.845098}{0.30103} = \boxed{2.807}$$

SUMMARY:

$$\textcircled{1} \quad \log_b 1 = \frac{0}{b > 0 \ (b \neq 1)} \quad \textcircled{2} \quad \log_b b = 1$$

$$\textcircled{3} \quad \log_c(ab) = \underline{\log_c a + \log_c b} \quad c > 0 \ (c \neq 1)$$

$$\textcircled{4} \quad \log_c\left(\frac{a}{b}\right) = \underline{\log_c a - \log_c b}$$

$$\textcircled{5} \quad \log_c a^n = n \log_c a$$

$$\textcircled{6} \quad \log_b a = \boxed{\frac{\log_x a}{\log_x b}} \quad \underline{x > 0 \ (x \neq 1)}$$

eg7: Write each of the following in terms of $\log 3$ and/or $\log 5$ and/or an integer:

a) $\log 45$

$$\log 45 = \log(9 \cdot 5)$$

$$\begin{aligned}\text{rule 3} \quad &= \log 9 + \log 5 \\ &= \log 3^2 + \log 5\end{aligned}$$

$$\text{rule 5} \quad = 2 \log 3 + \log 5$$

b) $\log \frac{25}{27}$

$$\begin{aligned}\log \frac{25}{27} &= \log 25 - \log 27 \\ \text{rule 4} \rightarrow &= \log 5^2 - \log 3^3\end{aligned}$$

$$\text{rule 5} \quad = 2 \log 5 - 3 \log 3$$

eg8: Find the exact value (no decimals) of the following:

a) $\log_7 \sqrt[4]{7}$

$$= \log_7 (7)^{\frac{1}{4}}$$

$$= \frac{1}{4} \log_7 7$$

$$= \frac{1}{4} (1)$$

$$= \boxed{\frac{1}{4}}$$

b) $\log_5 5^6 - \log_5 5^2$

$$= 6 \log_5 5 - 2 \log_5 5$$

$$= 4 \log_5 5$$

$$= 4 (1)$$

$$= \boxed{4}$$

or

$$\log_5 \left(\frac{5^6}{5^2} \right) = \log_5 (5^4)$$

$$= 4 \log_5 5$$

$$= 4 (1)$$

$$= \boxed{4}$$

$$\begin{aligned}
 c) \log_{\frac{1}{4}} \left(\frac{16^2}{2^{-3}} \right) &\rightarrow = \frac{\log 2^8}{\log 2^{-2}} \\
 &= \log_{\frac{1}{4}} \frac{(2^4)^2}{2^{-3}} \\
 &= \log_{\frac{1}{4}} \frac{2^8}{2^{-3}} \\
 &= \log_{\frac{1}{4}} 2^8 \\
 &= \frac{\log 2^8}{\log \frac{1}{4}} \quad \frac{1}{4} = 2^{-2} \\
 &= \boxed{\frac{-8}{2}} \\
 &= -4
 \end{aligned}$$

Rewriting Logarithmic Expressions

eg9: Expand each:

a) $\log_3 3x^4y^2$

$$= \log_3 3 + \log_3 x^4 + \log_3 y^2$$

$$= 1 + 4 \log_3 x + 2 \log_3 y$$

b) $\log \frac{\sqrt{2x-5}}{3}$

$$= \log \frac{(2x-5)^{\frac{1}{2}}}{3}$$

$$= \log (2x-5)^{\frac{1}{2}} - \log 3$$

$$= \boxed{\frac{1}{2} \log (2x-5) - \log 3}$$

Ex 10: Condense each to a log of a single quantity

a) $\frac{1}{3} \log x + 2 \log(x-1)$

$$= \log x^{\frac{1}{3}} + \log(x-1)^2$$

$$= \boxed{\log(\sqrt[3]{x} \cdot (x-1)^2)}$$

=
or

$$\boxed{\log((\sqrt[3]{x})^2 - 2x\sqrt[3]{x} + \sqrt[3]{x})}$$

b) $2 \log_3(x+4) - \log_3 x$

$$= \log_3(x+4)^2 - \log_3 x$$

$$= \boxed{\log_3 \frac{(x+4)^2}{x}}$$

c) $\log 5 + 2 \log x - 3 \log(x^2+5)$

$$= \log 5 + \log x^2 - \log(x^2+5)^3$$

$$= \boxed{\log \left(\frac{5x^2}{(x^2+5)^3} \right)}$$

$$\text{eg 11: Simplify } \frac{1}{\log_2 10} + \frac{1}{\log_5 10}$$

$$\begin{aligned}
 &= \frac{1}{\left(\frac{\log 10}{\log 2}\right)} + \frac{1}{\left(\frac{\log 10}{\log 5}\right)} \\
 &= \frac{\log 2}{\log 10} + \frac{\log 5}{\log 10} \\
 &= \frac{\log 2 + \log 5}{\log 10} \\
 &= \frac{\log(2 \cdot 5)}{\log 10} = \frac{\log 10}{\log 10} = \boxed{1}
 \end{aligned}$$

$$\text{eg 12: Simplify } 6 \log_9 x - 12 \log_{27} x$$

$$\begin{aligned}
 &= 6 \log_9 x^6 - 12 \log_{27} x^{12} \\
 &= \frac{6 \log x^6}{\log 9} - \frac{12 \log x^{12}}{\log 27} \\
 &= \frac{6 \log x}{\log 3^2} - \frac{12 \log x}{\log 3^3} \\
 &= \frac{6 \log x}{2 \log 3} - \frac{12 \log x}{3 \log 3} \quad \text{or} \quad \frac{3 \log x}{\log 3} - \frac{4 \log x}{\log 3} \\
 &= 3 \log_3 x - 4 \log_3 x
 \end{aligned}$$

$$= -\log_3 x$$

$$\begin{aligned}
 &= -\log_3 x
 \end{aligned}$$

Another base does exist on your calculator (other than 10), and that is $e = \underline{2.718282\dots}$

$$y = \log x = \log_{10} x \quad \text{common log fn}$$

$$y = \ln x = \log_e x \quad \text{natural log fn}$$

eg 13: Evaluate to 3 decimal places using \ln :

$$\log_6 532$$

$$= \frac{\log_e 532}{\log_e 6}$$

$$= \frac{\ln 532}{\ln 6} = \frac{6.2766}{1.79176} = \boxed{3.503}$$

* another useful Rule:

$$\boxed{b^{\log_b a} = a \quad (a > 0)}$$

since

$$b^{\log_b a} = y$$

$$\log_b a = \log_b y$$

$$\boxed{y = a}$$

eg 14: Simplify $x^{\log_x 11}$

Using rule

$$= \boxed{11}$$

Long way:

$$x^{\log_x 11} = y$$

$$\log_x(x^{\log_x 11}) = \log_x y$$

$$\log_x 11 \log_x x = \log_x y$$

$$\log_x 11 = \log_x y$$

$$\boxed{y = 11}$$

Three common mistakes with respect to
the Rules:

- i) $\log(a+b) \neq \underline{\log a + \log b}$
- ii) $(\log a)^2 \neq \underline{2 \log a}$
- iii) $\frac{\log a}{\log b} \neq \underline{\log a - \log b}$

Also, always check solutions for extraneous
roots (more on this next section)

p. 221 # 1-5

* adjust instructions for Q#1.

typo lg p 437

Ch. 5.4 - Solving Exponential and Logarithmic Equations

Remember: $\Rightarrow a^x = a^y$ if and only if $x=y$;
 $\Rightarrow \log_a x = \log_a y$ if and only if $x=y$.

Also, you must ALWAYS check solutions for EXTRANEous ROOTS. This, because in $\log_a x$, $a > 0 (a \neq 1)$ and $x > 0$.

It is absolutely necessary to employ previously learned skills such as:

- converting equations from logarithmic to exponential form and,
- vice versa.

eg1: Solve the following for x using logs:

a) $2^x = 8$

$$\log 2^x = \log 8 \quad * \text{rule 2}_{\text{above}}$$

$$x \log 2 = \log 8$$

$$x = \frac{\log 8}{\log 2} = \frac{\log 2^3}{\log 2}$$

$$x = \frac{3 \log 2}{\log 2}$$

$$\boxed{x = 3} \quad \text{Check!}$$

b) $3^x = 11$

$$\log 3^x = \log 11$$

$$x \log 3 = \log 11$$

$$x = \frac{\log 11}{\log 3}$$

$$\boxed{x = 2.18}$$

} need calculator!

Check!

eq 2: Solve for x :

$$\log(x+3) + \log x = 1$$

$$\log(x(x+3)) = 1$$

Product
Property

$$x(x+3) = 10^1$$

log \rightarrow exp. form

$$x^2 + 3x - 10 = 0$$

$$(x+5)(x-2) = 0$$

$$x = -\cancel{5}, \boxed{2}$$

$$x > 0 \text{ in } \log x$$

eq 3: Solve for x :

$$\log_3(x+6) - \log_3(x+2) = \log_3 x$$

$$\log_3 \left[\frac{(x+6)}{(x+2)} \right] = \log_3 x$$

$$3^{\log_3 x} = \frac{x+6}{x+2}$$

or

$$\frac{x+6}{x+2} = x$$

$$3^{\log_3 \left(\frac{x+6}{x+2} \right)} = x$$

$$x+6 = x^2 + 2x$$

$$0 = x^2 + x - 6$$

$$(x+3)(x-2) = 0$$

$$x = -\cancel{3}, \boxed{2} \text{ Check!}$$

Change of base:

$$\frac{\log \frac{8}{4}}{\log 3} = \frac{\log 2}{\log 3}$$

eg 4: Solve for x :

$$2 \log_3 x + \log_3 (x-1) = 1 + \log_3 2x$$

$$\log_3 x^2 + \log_3 (x-1) = 1 + \log_3 2x$$

$$\log_3 [x^2(x-1)] = 1 + \log_3 2x$$

$$\log_3 (x^2(x-1)) - \log_3 2x = 1$$

$$\log_3 \left[\frac{(x^2(x-1))}{2x} \right] = 1$$

$$\log_3 \left[\frac{x(x-1)}{2} \right] = 1$$

$$x^3 - x^2 = 6x$$

$$x^3 - x^2 - 6x = 0$$

$$\frac{x^2 - x}{2} = 3^1$$

$$x^2 - x = 6$$

$$x^2 - x - 6 = 0$$

$$(x-3)(x+2) = 0$$

$$x = \boxed{3}, -\cancel{2}, \cancel{0}$$

Check! $2 \log_3 3 + \log_3 2 = 1 + \log_3 6$

$$\log_3 9 + \log_3 2 = 1 + \log_3 6$$

$$\log_3 18 = 1 + \log_3 6$$

$$\log_3 3 = 1 \quad \checkmark$$

eg 5: Solve for x :

$$x^{\log x} = 100x$$

$$\log_x 100x = \log x$$

$$\log x^{\log x} = \log 100x$$

$$\text{or } \frac{\log 100x}{\log x} = \log x$$

$$\log x \log x = \log 100x$$

$$\log 100 + \log x = (\log x)^2$$

$$2 + \log x = (\log x)^2$$

$$(\log x)^2 = \log 100x$$

$$(\log x)^2 - \log x - 2 = 0$$

$$(\log x)^2 = \log 100 + \log x$$

$$(\log x)^2 - \log x - 2 = 0 \quad \checkmark$$

$$(\log x - 2)(\log x + 1) = 0$$

$$\log x = 2 \quad \log x = -1$$

$$x = 10^2$$

$$x = 10^{-1}$$

$$x = 100$$

$$x = \frac{1}{10} \quad \checkmark \text{ check!}$$

$$\left(\frac{1}{10}\right)^{-1} = 100 \left(\frac{1}{10}\right)$$

$$10 = 10 \quad \checkmark$$

eg 6: Solve $3 \cdot 2^{x-2} = 6^x$ in terms of logarithms:

$$\log(3 \cdot 2^{x-2}) = \log 6^x$$

$$\log 3 + \log 2^{x-2} = \log 6^x$$

$$\log 3 + (x-2) \log 2 = x \log 6$$

$$\log 3 + x \log 2 - 2 \log 2 = x \log 6$$

$$x \log 2 - x \log 6 = 2 \log 2 - \log 3$$

$$x(\log 2 - \log 6) = \log 2^2 - \log 3$$

$$x = \frac{\log 4 - \log 3}{\log 2 - \log 6}$$

$$x = \frac{\log(\frac{4}{3})}{\log(\frac{1}{3})}$$

OR

$$\left\{ \begin{array}{l} \frac{\log \frac{4}{3}}{-\log 3} \\ \text{OR} \end{array} \right.$$

$$\frac{-\log \frac{4}{3}}{\log 3}$$

OR

$$\frac{\log \frac{3}{4}}{\log 3}$$

eg 7: Solve for A in terms of B and C:

$$2 \log A - \log B = C$$

$$\log A^2 - \log B = C$$

$$\log\left(\frac{A^2}{B}\right) = C$$

$$\frac{A^2}{B} = 10^C$$

$$A^2 = B \cdot 10^C$$

$$A = \sqrt{B \cdot 10^C} = \boxed{\sqrt{B \cdot 10^C}}$$

eg8: If $\log 3 = a$ and $\log 8 = b$, determine $\log 18$ in terms of a and b .

$$\begin{aligned}
 \log 18 &= \log 9 \cdot 2 \\
 &= \log 9 + \log 2 \\
 &= \log 3^2 + \log 2 \\
 &= 2 \log 3 + \log 2 \\
 &= 2a + \log 8^{\frac{1}{3}} \\
 &= 2a + \frac{1}{3} \log 8 \\
 &= 2a + \boxed{\frac{b}{3}}
 \end{aligned}$$

Typo: 1 if p.440
1 if p.439, 2 h if p.440

p. 227 i.e. # 1-7 (8 for fun) (2h, if for fun)

eg9: If $\log 3 = a$ and $\log 5 = b$, what is $\log_2 45$ in terms of a and b and an integer?

$$\begin{aligned}
 \log_2 45 &= \frac{\log 45}{\log 2} = \frac{\log (9 \cdot 5)}{\log 2} = \frac{\log 9 + \log 5}{\log 2} \\
 &= \frac{2 \log 3 + \log 5}{\log 2} = \frac{2a + b}{\log 2} = \frac{2a + b}{\log (\frac{10}{5})} \\
 &= \frac{2a + b}{\log 10 - \log 5} = \boxed{\frac{2a + b}{1 - b}}
 \end{aligned}$$

Ch. 5.5 - Applications of Exponential and Log Fns

i.e. ... more word problems!

Derivation of the Compound Interest Formula:

$$A = P \left(1 + \frac{r}{n}\right)^{nt}$$

- suppose P dollars is deposited at rate r for one year

$$\text{Interest after one year} = I_1 = P \cdot r \cdot 1 = Pr$$

$$\begin{aligned}\text{Amount after one year (total)} &= A_1 = P + I_1 = P + Pr \\ &= P(1+r)\end{aligned}$$

I_2 is based upon A_1 (not A_0 or P):

$$I_2 = [P(1+r) \cdot r] \quad \text{factoring out } P(1+r)$$

$$\begin{aligned}\text{so, } A_2 &= P(1+r) + P(1+r)r = P(1+r)(1+r) \\ &= P(1+r)^2\end{aligned}$$

$$I_3 = [P(1+r)^2 \cdot r]$$

$$\begin{aligned}A_3 &= P(1+r)^2 + P(1+r)^2 r \\ &= P(1+r)^2 (1+r) \quad \text{factor out } P(1+r)^2\end{aligned}$$

$$= P(1+r)^3 * \text{see the pattern?}$$

* t years: $A_t = P(1+r)^t$

... if compounded annually
(once per year)

- to calculate more frequent compoundings (eg: quarterly, monthly, daily etc...), let n be the number of compounds per year and t the total number of years.
- r is an annual rate, so the rate per compound would be $\frac{r}{n}$ for t years.
- the total number of compounds would be nt .

$$A = P \left(1 + \frac{r}{n}\right)^{nt}$$

Interestingly, e ($2.71828\dots$) shows up in the following manner:

Say $P = \$1$ for 1 year at nominal rate of 100%

$$A = \left(1 + \frac{1}{n}\right)^{n(1)} = \left(1 + \frac{1}{n}\right)^n$$

- as n grows larger and larger, $A = \left(1 + \frac{1}{n}\right)^n$ approaches e .

* see table on p. 189 for numerical proof

COMPOUND INTEREST FORMULAS

$$A = P \left(1 + \frac{r}{n}\right)^{nt}$$

* interest compounded
 n times per year

$$A = Pe^{rt}$$

* interest compounded
CONTINUOUSLY (compounding period infinitely small)

Eg 1: Estimate the time required for \$5000 to grow to \$30000 if it is invested at 10% compounded:

a) monthly (nearest hundredth) b) continuously

$$A = P \left(1 + \frac{r}{n}\right)^{nt}$$

$$30000 = 5000 \left(1 + \frac{0.1}{12}\right)^{12t}$$

$$6 = \left(1 + \frac{0.1}{12}\right)^{12t}$$

$$\log 6 = 12t \log \left(1 + \frac{0.1}{12}\right)$$

$$t = \frac{\log 6}{12 \log \left(1 + \frac{0.1}{12}\right)}$$

$$= 17.99 \text{ years}$$

$$A = Pe^{rt}$$

$$30000 = 5000 e^{(0.1)t}$$

$$6 = e^{0.1t}$$

$$\ln 6 = 0.1t \ln e$$

$$\ln 6 = 0.1t$$

$$t = \frac{\ln 6}{0.1}$$

$$t = 17.92 \text{ years}$$

Recall: Growth and Decay Formulas

$$A = A_0 (x)^{\frac{t}{T}}$$

A = final amt.

A_0 = initial amt.

x = growth or decay value

eg: half-life $\Rightarrow 0.5$

incr. by 15% $\Rightarrow 1.15$

decr. by 25% $\Rightarrow 0.75$

t = total time that item has grown/existed

T = time required by A_0 to incr./decr. by a factor of x

$$A = A_0 e^{kt}$$

e = constant = 2.71828...

k = proportional constant

t = time

if not provided,
will have to
be calculated

eg2: The half-life of Plutonium-241 is 13 years. Find the time it takes for 80% of a 10 gram sample to decay. Round to the nearest hundredth of a year. Use both methods.

Note: 80% of 10g \rightarrow $0.80 \times 10\text{g} = 8\text{g}$ decays
 $10\text{g} - 8\text{g} = 2\text{g}$ remaining

Method 1

$$A = A_0 (x)^{\frac{t}{T}}$$

$$2 = 10 (0.5)^{\frac{t}{13}}$$

$$0.2 = 0.5^{\frac{t}{13}}$$

$$\log 0.2 = \log 0.5^{\frac{t}{13}}$$

$$\log 0.2 = \frac{t}{13} \log 0.5$$

$$t = \frac{13 \log 0.2}{\log 0.5}$$

$$t = 30.19 \text{ yrs}$$

Method 2

$$A = A_0 e^{kt}$$

To find k:

$$A_0 (x)^{\frac{t}{T}} = A_0 e^{kt}$$

$$x^{\frac{t}{T}} = e^{kt}$$

$$x^{\frac{t}{T}} = e^k$$

$$k = \ln x^{\frac{t}{T}}$$

$$k = \left(\frac{1}{T}\right) \ln x$$

$$k = \frac{\ln x}{T}$$

$$2 = 10 e^{(\frac{\ln 0.5}{13})t}$$

$$0.2 = e^{(\frac{\ln 0.5}{13})t}$$

$$\frac{t \ln 0.5}{13} = \ln 0.2$$

$$t = \frac{13 \ln 0.2}{\ln 0.5} = 30.19 \text{ yrs}$$

eg3: In a research experiment, a population of fruit flies increases according to the law of exponential growth. After 2 days there are 100 flies. After 4 days, there are 300 flies. How many flies will there be after 9 days? Round to the nearest fly and use both methods.

HINT: find x for $T = 1$ day.

Method 1

$$A = A_0 (x)^{\frac{t}{T}}$$

$$300 = 100 (x)^{\frac{2}{1}}$$

$$3 = x^2$$

$$x = \pm \sqrt{3}$$

$$x = \sqrt{3}$$

so,

$$A = A_0 (x)^{\frac{t}{T}}$$

$$A = 100 (\sqrt{3})^{\frac{7}{1}}$$

$$A = 4677 \text{ flies}$$

Method 2

$$A = A_0 e^{kt}$$

$$k = \frac{\ln x}{T}$$

$$k = \frac{\ln \sqrt{3}}{1} = \ln \sqrt{3}$$

$$A = 100 e^{(\ln \sqrt{3})(7)}$$

$$A = 4677 \text{ flies}$$

eg4: A restaurant is preparing a turkey.

At 12 noon, the internal temperature of the turkey was 75°F . At 2:30 pm, the turkey's temperature was 105°F . Assuming that the oven temperature remained constant, when was the turkey ready to eat if its temp. needed to be 175°F ? Round to the nearest minute.

Again, find x for $T = 1$ hour.

$$A = A_0 (x)^{\frac{t}{T}}$$

$$105 = 75 (x)^{\frac{2.5}{1}}$$

$$\frac{105}{75} = x^{2.5}$$

$$x = 1.144$$

so,

$$A = A_0 (x)^{\frac{t}{T}}$$

$$175 = 75 (1.144)^{\frac{t}{1}}$$

$$2.3 = 1.144^t$$

$$t = \log_{1.144} 2.3$$

$$t = 6.30 \text{ hrs.}$$

$$A = A_0 e^{kt}$$

$$k = \frac{\ln 1.144}{1} = 0.13453$$

$$175 = 75 e^{(0.13453)t}$$

$$2.3 = e^{0.13453 t}$$

$$0.13453 t = \ln 2.3$$

$$t = \frac{\ln 2.3}{0.13453}$$

$$t = 6.30 \text{ hrs}$$

$$\frac{0.30 \text{ h}}{1 \text{ h}} \left| \frac{60 \text{ min.}}{} \right. = 18 \text{ mins.}$$

6:18 pm

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