

Ch. 6.1 - Trig. Functions - Angles and their Measures

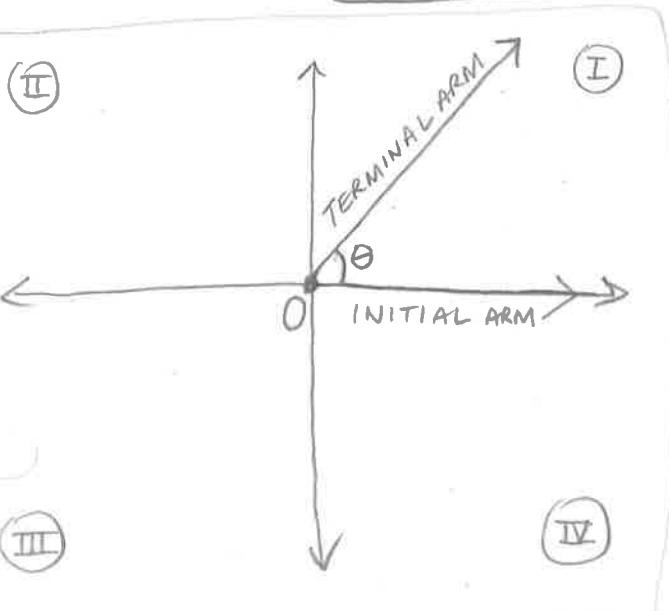
Angles on a Coordinate Plane in STANDARD POSITION:

- standard position implies that the angle's INITIAL ARM rests along the positive x-axis of the coordinate plane, with its vertex at the origin.

It further implies that the initial arm is rotated about the origin (vertex) which creates angle θ .

The final 'resting place' of the arm is referred to as the angle's

TERMINAL ARM.



note:

- a counter-clockwise rotation of the terminal arm creates a POSITIVE $\angle \theta$.
- a clockwise rotation of the terminal arm creates a NEGATIVE $\angle \theta$.

- the measure of $\angle \theta$, in standard position, is governed by the DIRECTION of the rotation and the MAGNITUDE (amount) of the rotation.
- amount of rotation often measured in DEGREES.
 (where one COMPLETE rotation is equivalent to $\pm 360^\circ$).

One DEGREE (1°) = $\boxed{\frac{1}{360}}$ of a complete rotation.

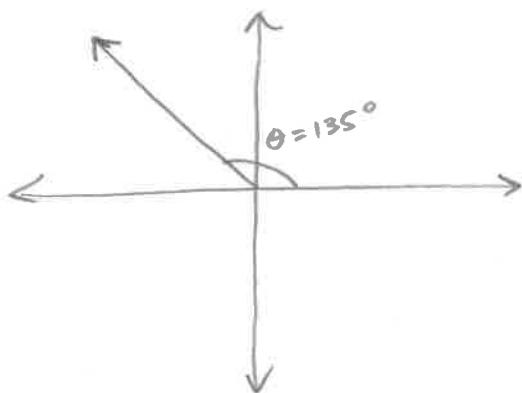
e.g.: Describe, in words, the direction and magnitude of each of the following angle measures:

a) $2^\circ = \frac{2}{360} = \frac{1}{180}$ of a complete counter-clockwise rotation

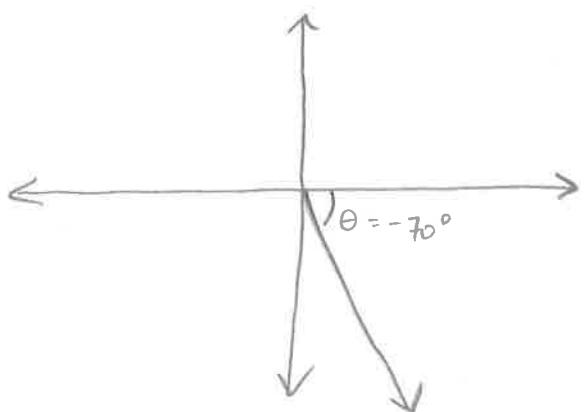
b) $-90^\circ = -\frac{90}{360} = -\frac{1}{4} = \frac{1}{4}$ of a complete clockwise rotation

eg 2: Sketch each of the following in standard position:

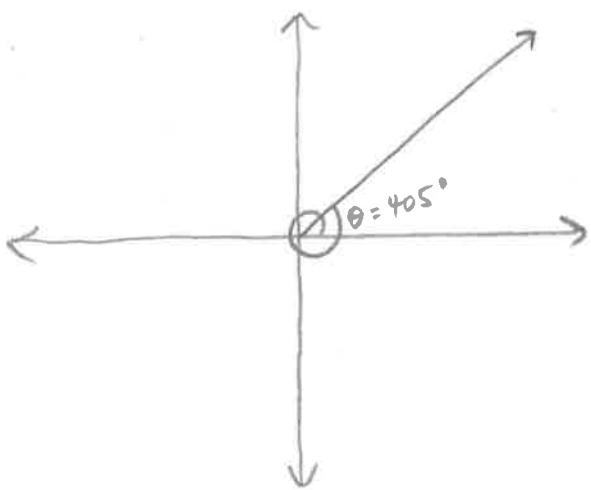
a) $\theta = 135^\circ$



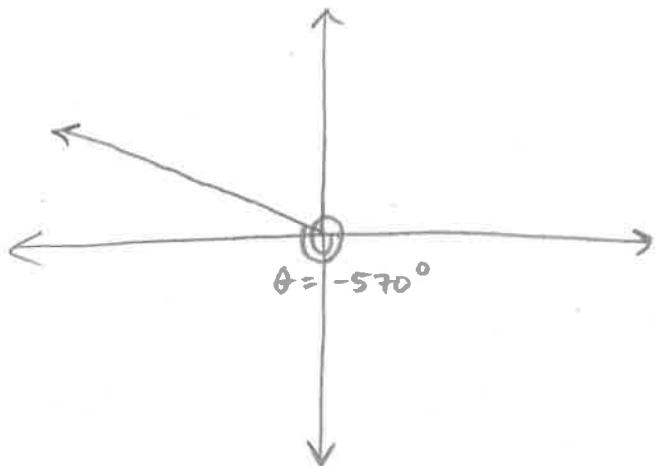
b) $\theta = -70^\circ$



c) $\theta = 405^\circ$



d) $\theta = -570^\circ$



Classes of Angles

Acute L: $0^\circ < \theta < 90^\circ$

Reflex L: $180^\circ < \theta < 360^\circ$

Right L: $\theta = 90^\circ$

Quadrantal Ls:

Obtuse L: $90^\circ < \theta < 180^\circ$

$\theta = 0^\circ, 90^\circ, 180^\circ, 270^\circ,$

Straight L: $\theta = 180^\circ$

$360^\circ, 450^\circ, \text{etc...}$

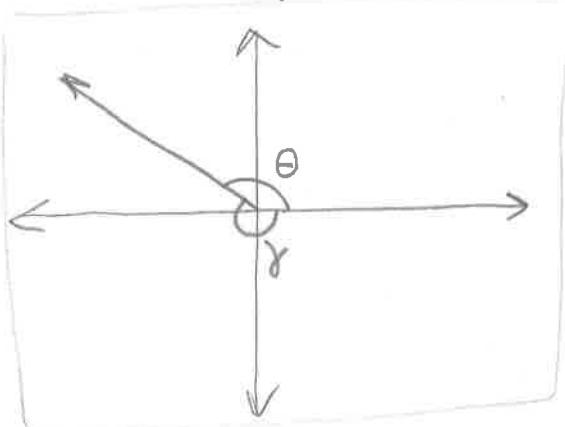
Co-terminal Angles

Co-terminal angles are angles that share the same terminal arm.

To find a co-terminal angle, add or subtract 360° to a given $\angle \theta$.

note: there exist infinitely many co-terminal angles.

eg3: On the grid provided, sketch $\theta = 150^\circ$ and $\gamma = -210^\circ$. What conclusion can you draw?



Conclusion: θ and γ are co-terminal angles.

eg4: Given $\theta = 465^\circ$, determine the two smallest positive and the two 'smallest' negative co-terminal angles.

i) Positive: $465^\circ - 360^\circ = 105^\circ$
 $465^\circ + 360^\circ = 825^\circ$

ii) Negative:
 $465^\circ - 2(360^\circ) = -255^\circ$
 $465^\circ - 3(360^\circ) = -615^\circ$

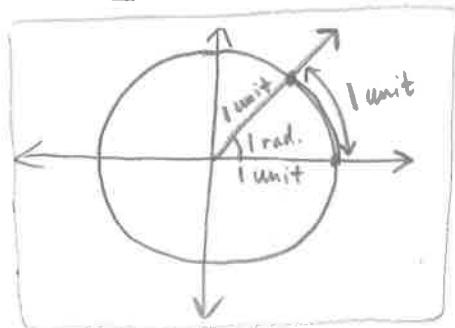
Radian Measure and Conversion

- aside from degrees, radians are another unit used to quantify an angle's magnitude.

Why another unit?

- radians are more suited to scientific/engineering applications since a radian also serves to directly define the length of an arc of a circle 'created' by angle θ . (note: this arc length is proportional to the radius of the circle).
- a **UNIT CIRCLE** (a circle with a radius of 1 unit) is used to define a radian:

An angle measuring 1 radian is a standard-position angle (counter-clockwise) that 'creates' an arc length of 1 on a UNIT circle.



$$\text{Circumference (circle)} = \underline{2\pi r}$$

$$\text{Circumference (unit circle)} = \underline{2\pi(1)} = 2\pi$$

So... $360^\circ = \underline{2\pi \text{ radians}}$

$\therefore 180^\circ = \underline{\pi \text{ radians}}$

note: there is no symbol for radians.

eg5: Convert each given degree value to radians:

a) 240°

$$\frac{240^\circ}{180^\circ} = \frac{4\pi}{3}$$
$$= \sim 4.2$$

b) 72°

$$\frac{72^\circ}{180^\circ} = \frac{2\pi}{5}$$
$$= \sim 1.3$$

eg6: Convert each given radian value to degrees:

a) $\frac{3\pi}{4}$

$$\frac{\frac{3\pi}{4}}{\pi} = 135^\circ$$

b) 2.13

$$\frac{2.13}{\pi} = 122^\circ$$

Key conversions to know:

$$\frac{\pi}{2} = \underline{90^\circ}$$

$$\frac{\pi}{4} = \underline{45^\circ}$$

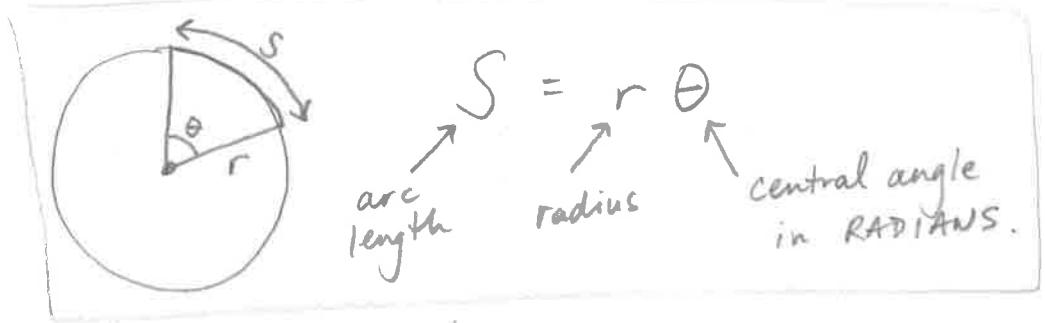
$$\frac{\pi}{6} = \underline{30^\circ}$$

$$\frac{\pi}{3} = \underline{60^\circ}$$

$$\frac{3\pi}{2} = \underline{270^\circ}$$

Arc length

The length of an arc of a circle (S) is directly proportional to the size of $\angle \theta$ (in radians) and the radius of the circle.



Eg 7: What is the arc length of an arc on a circle with $r = 5 \text{ cm}$ and $\theta = 60^\circ$?

$$S = r\theta$$

$$60^\circ = \frac{\pi}{3}$$

$$S = 5 \left(\frac{\pi}{3} \right)$$

$$S = \frac{5\pi}{3} \text{ cm}$$

eg8: Find θ if a circle has a diameter of 12 cm and has an arc (defined by θ) of 18 cm? Round answer to nearest degree.

$$S = r\theta$$

$$\theta = \frac{S}{r}$$

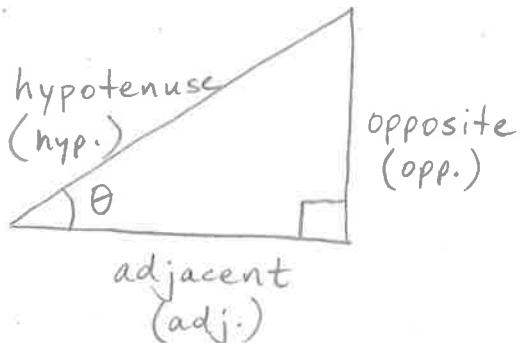
$$\theta = \frac{18}{6}$$

$$\theta = 3 \text{ radians} \quad \left| \begin{array}{l} 180^\circ \\ \hline \pi \end{array} \right. = \boxed{172^\circ}$$

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Ch. 6.2 - Trigonometric Functions of Acute Angles

- a trigonometric function is a ratio of two side lengths of a right triangle.



The 6 Trig. Functions

For an acute angle θ in a right triangle:

$$\text{SINE } \theta = \sin \theta = \frac{\text{opp.}}{\text{hyp.}}$$

(SOH)

$$\text{COSECANT } \theta = \csc \theta = \frac{\text{hyp.}}{\text{opp.}}$$

(CHo)

$$\text{COSINE } \theta = \cos \theta = \frac{\text{adj.}}{\text{hyp.}}$$

(CAH)

$$\text{SECANT } \theta = \sec \theta = \frac{\text{hyp.}}{\text{adj.}}$$

(SHA)

$$\text{TANGENT } \theta = \tan \theta = \frac{\text{opp.}}{\text{adj.}}$$

(TOA)

$$\text{COTANGENT } \theta = \cot \theta = \frac{\text{adj.}}{\text{opp.}}$$

(CAO)

Note: $\sin \theta = \frac{1}{\csc \theta}$

$$\csc \theta = \frac{1}{\sin \theta}$$

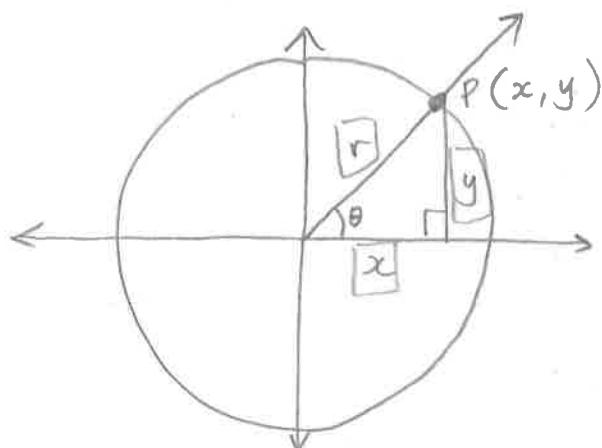
$$\cos \theta = \frac{1}{\sec \theta}$$

$$\sec \theta = \frac{1}{\cos \theta}$$

$$\tan \theta = \frac{1}{\cot \theta}$$

$$\cot \theta = \frac{1}{\tan \theta}$$

Consider angle θ in standard position with point $P(x, y)$ on the terminal side of θ :



CONSTRUCT A RIGHT Δ
BY CONNECTING P TO
THE NEAREST POINT ON
THE x -AXIS.

Using the Pythagorean Theorem:

$$x^2 + y^2 = r^2$$

$$r = \sqrt{x^2 + y^2}$$

Note: r is always positive.

We can use x , y , and r to define the 6 trig. ratios of any $\angle \theta$ (even non-acute)

↳ reference $\angle s$

$$\sin \theta = \frac{y}{r}$$

$$\csc \theta = \frac{r}{y}$$

$$\cos \theta = \frac{x}{r}$$

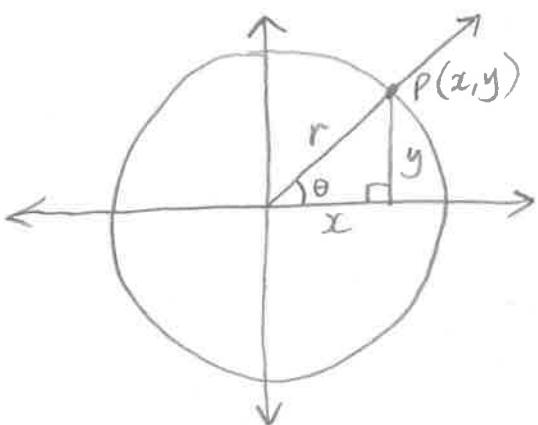
$$\sec \theta = \frac{r}{x}$$

$$\tan \theta = \frac{y}{x}$$

$$\cot \theta = \frac{x}{y}$$

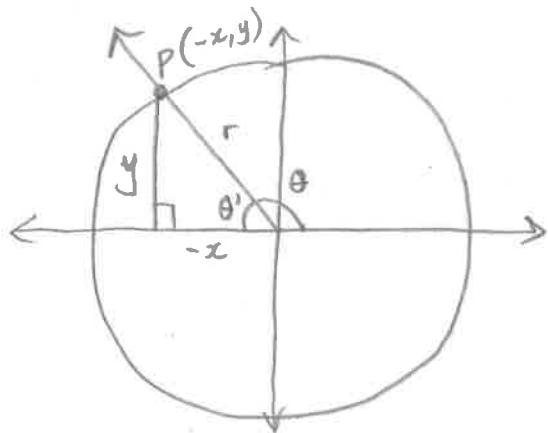
θ may exist in any quadrant!

i) θ in Quadrant I:



* all 6 trig. ratios positive

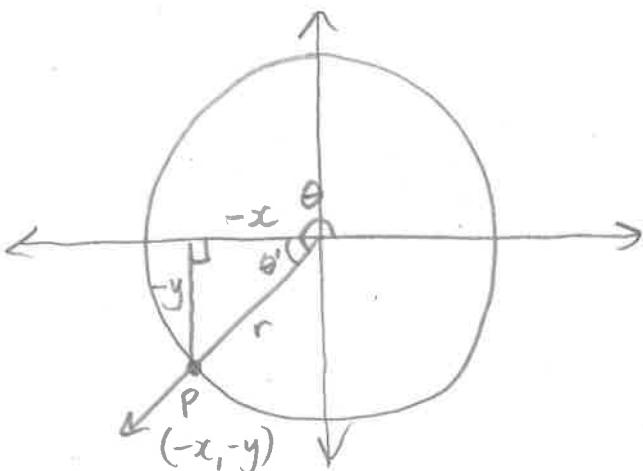
ii) θ in Quadrant II:



* $\cos \theta, \sec \theta, \tan \theta, \cot \theta$
all negative.

* $\sin \theta, \csc \theta$ both positive.

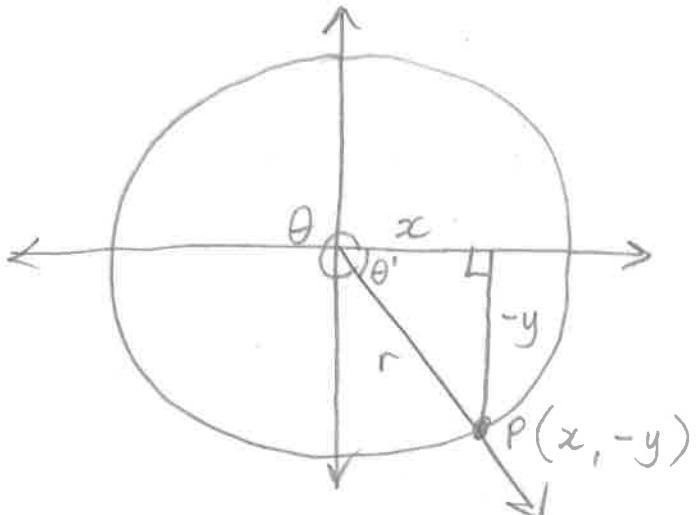
iii) θ in Quadrant III:



* $\sin \theta, \csc \theta, \cos \theta, \sec \theta$
all negative

* $\tan \theta, \cot \theta$ both positive

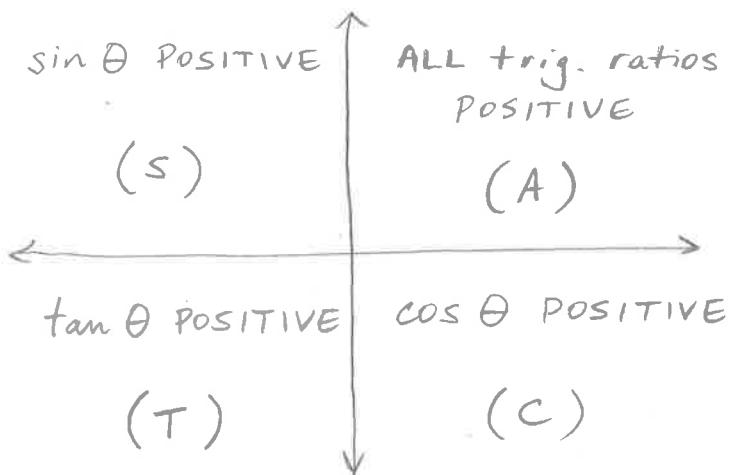
iv) θ in Quadrant IV:



* $\sin \theta, \csc \theta, \tan \theta, \cot \theta$
all negative

* $\cos \theta, \sec \theta$ both positive.

Summary:



* if $\sin \theta$ is positive, then so too is its reciprocal function, $\csc \theta$, and so on for $\cos \theta$ and $\tan \theta$.

eg1: Which quadrant does θ exist in standard position if $\sin \theta < 0$ and $\tan \theta > 0$?

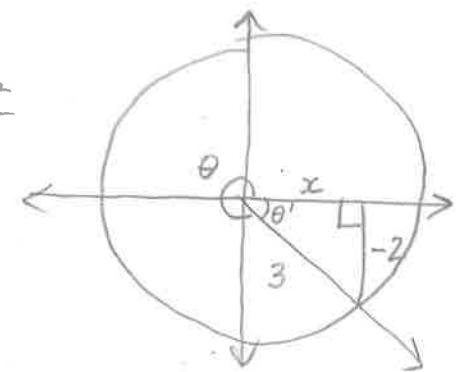
$\sin \theta < 0$ in $\varphi \text{ III}$ and $\varphi \text{ IV}$

$\tan \theta > 0$ in $\varphi \text{ I}$ and $\varphi \text{ III}$

} θ in $\varphi \text{ III}$

eg2: Determine $\cos \theta$ if $\csc \theta = -\frac{3}{2}$ and $\tan \theta < 0$.

$\csc \theta < 0$ in $\varphi \text{ III}, \text{IV}$
 $\tan \theta < 0$ in $\varphi \text{ II}, \text{IV}$



$$a^2 + b^2 = c^2$$

$$x^2 + (-2)^2 = 3^2$$

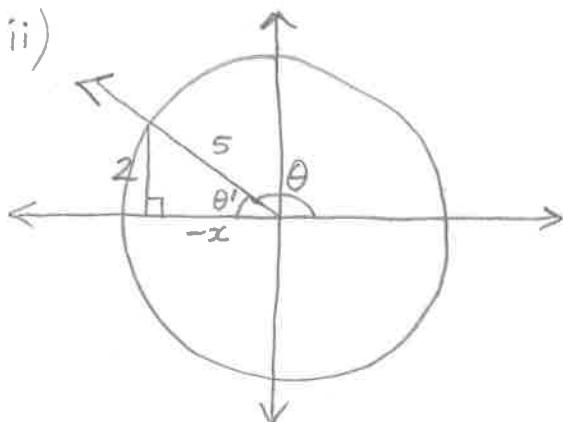
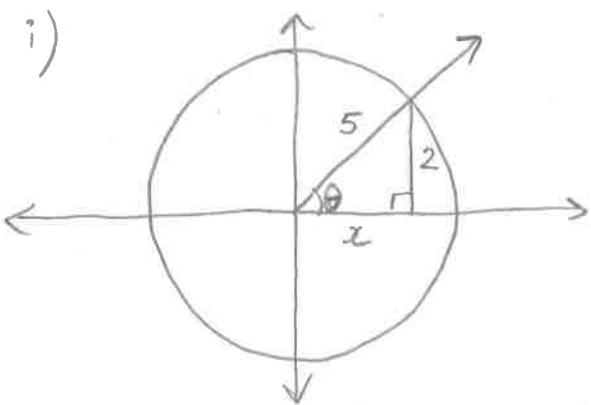
$$x^2 = 5$$

$$x = \pm \sqrt{5} = \sqrt{5}$$

$$\cos \theta = \frac{\sqrt{5}}{3}$$

Eg 3: Determine $\cot \theta$ if $\sin \theta = \frac{2}{5}$.

$\sin \theta > 0$ in Q I, II



$$a^2 + b^2 = c^2$$

$$x^2 + 2^2 = 5^2$$

$$x^2 = 21$$

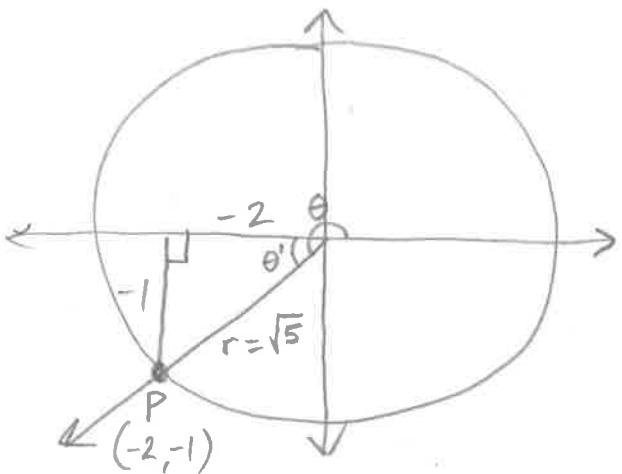
$$x = \pm \sqrt{21}$$

$$x = \sqrt{21} \text{ in Q I}$$

$$x = -\sqrt{21} \text{ in Q II}$$

$$\left. \begin{array}{l} \cot \theta = \frac{\sqrt{21}}{2}, \quad -\frac{\sqrt{21}}{2} \\ (\text{I}) \quad (\text{II}) \end{array} \right\}$$

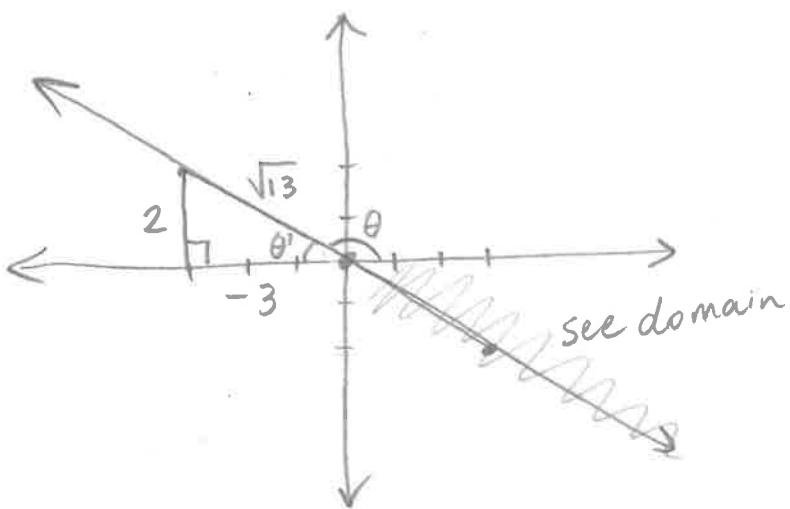
Eg 4: Given the point $P(-2, -1)$ on the terminal side of $\angle \theta$ in standard position, determine the value of all 6 trig. functions.



$$\begin{aligned} a^2 + b^2 &= c^2 \\ (-2)^2 + (-1)^2 &= r^2 \rightarrow r = \sqrt{5} \end{aligned}$$

$\sin \theta = \frac{-1}{\sqrt{5}} = -\frac{\sqrt{5}}{5}$	$\csc \theta = \frac{\sqrt{5}}{-1} = -\sqrt{5}$
$\cos \theta = \frac{-2}{\sqrt{5}} = -\frac{2\sqrt{5}}{5}$	$\sec \theta = \frac{\sqrt{5}}{-2} = -\frac{\sqrt{5}}{2}$
$\tan \theta = \frac{-1}{-2} = \frac{1}{2}$	$\cot \theta = \frac{-2}{-1} = 2$

Eg5: Determine $\sin \theta$ and $\cos \theta$ if θ is an angle in standard position whose terminal side is the line $2x + 3y = 0$ ($x \leq 0$).



$$2x + 3y = 0$$

$$3y = -2x$$

$$y = -\frac{2}{3}x$$

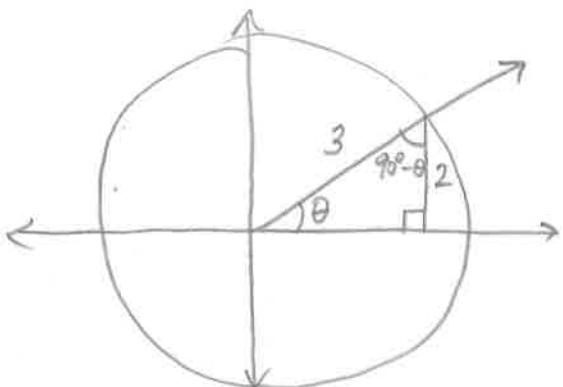
$$a^2 + b^2 = c^2$$

$$(-3)^2 + (2)^2 = r^2$$

$$r = \sqrt{13}$$

$$\boxed{\begin{aligned}\sin \theta &= \frac{2}{\sqrt{13}} = \frac{2\sqrt{13}}{13} \\ \cos \theta &= \frac{-3}{\sqrt{13}} = \frac{-3\sqrt{13}}{13}\end{aligned}}$$

Eg6: If $\sin \theta = \frac{2}{3}$, find $\cos(90^\circ - \theta)$.



$$\cos(90^\circ - \theta) = \boxed{\frac{2}{3}}$$

Note: $\sin \theta = \underline{\cos(90^\circ - \theta)}$ and,

$$\cos \theta = \underline{\sin(90^\circ - \theta)}$$

$$\sec \theta = \underline{\csc(90^\circ - \theta)}$$

$$\csc \theta = \underline{\sec(90^\circ - \theta)}$$

$$\tan \theta = \underline{\cot(90^\circ - \theta)}$$

$$\cot \theta = \underline{\tan(90^\circ - \theta)}$$

Eg7: Find the smallest positive angle θ such that $\sin \theta = \cos \theta$.

$$\sin \theta = \cos \theta$$

also: $\sin \theta = \cos(90^\circ - \theta)$

so, $\cos \theta = \cos(90^\circ - \theta)$

thus, $\theta = 90^\circ - \theta$

$$2\theta = 90^\circ$$

$$\boxed{\theta = 45^\circ}$$

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(14, 15 for 'fun')

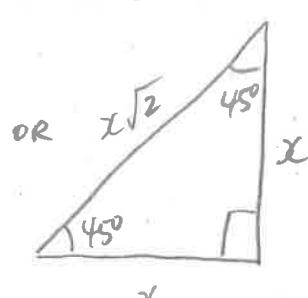
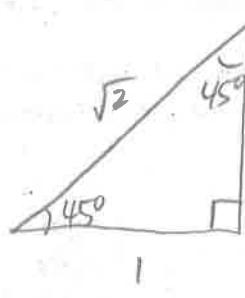
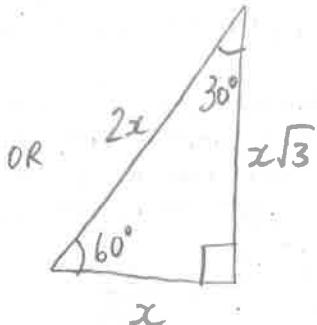
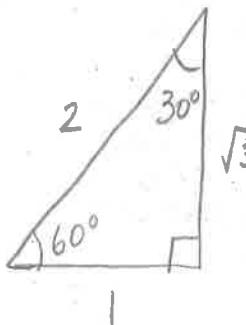
Ch. 6.3 - Trig Functions - General and Special Angles

Special Angles

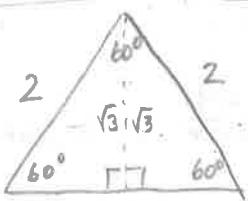
→ 30° , 45° , and 60° or $\frac{\pi}{6}$, $\frac{\pi}{4}$, and $\frac{\pi}{3}$

- derived from special triangles:

$30^\circ - 60^\circ - 90^\circ \Delta$

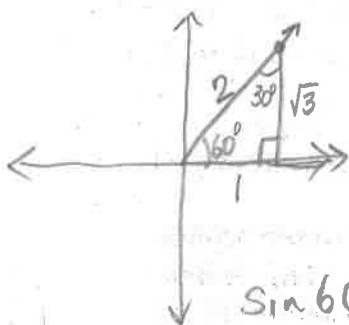


Since:



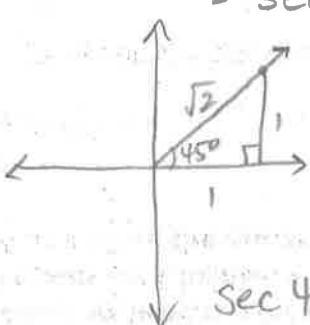
e.g.: Evaluate the following (include a diagram of the angle in standard position):

a) $\sin 60^\circ$



$$\sin 60^\circ = \boxed{\frac{\sqrt{3}}{2}}$$

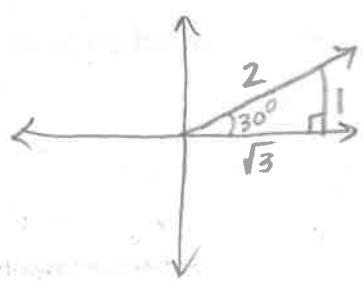
b) $\sec \frac{\pi}{4}$



$$= \sec 45^\circ$$

c) $\tan \frac{\pi}{6}$

$$= \tan 30^\circ$$



$$\tan \frac{\pi}{6} = \boxed{\frac{1}{\sqrt{3}}}$$

$$\frac{1}{\cos 45^\circ} = \frac{1}{(\frac{1}{\sqrt{2}})}$$
$$\boxed{= \sqrt{2}}$$

$$= \boxed{\frac{\sqrt{3}}{3}}$$

Using your diagrams from example 1, fill in the following table:

θ	$\sin \theta$	$\cos \theta$	$\tan \theta$	$\csc \theta$	$\sec \theta$	$\cot \theta$
$30^\circ = \frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$	2	$\frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$	$\sqrt{3}$
$45^\circ = \frac{\pi}{4}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
$60^\circ = \frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$	2	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

* notice that all trig. ratios in the table are positive since all three θ values are in QI.

Reference Angles

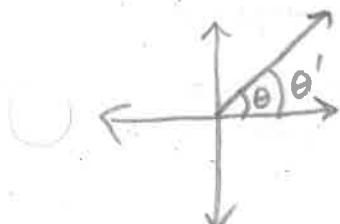
Def'n: For angle θ in standard position, the

REFERENCE ANGLE is the positive, acute angle θ' that is formed between the terminal side of θ and the x-axis.

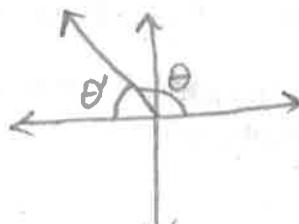
$$0^\circ < \theta' < 90^\circ \text{ or } 0 < \theta' < \frac{\pi}{2}$$

- a reference angle 'represents' θ when θ is too large to fit inside a right Δ .

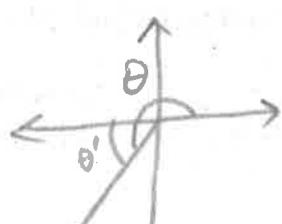
QI:



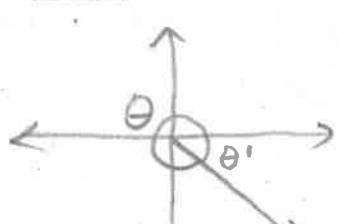
QII:



QIII:



QIV:



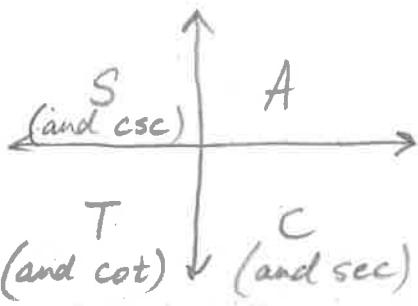
$$\theta' = \theta$$

$$\theta' = 180^\circ - \theta$$

$$\theta' = \theta - 180^\circ$$

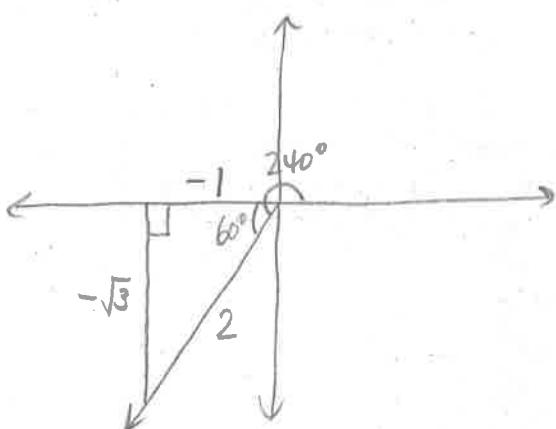
$$\theta' = 360^\circ - \theta$$

Also, recall



eg2: Determine the exact value of the following: (diagrams required)

a) $\sin 240^\circ$



$$\theta' = 60^\circ \text{ (in QIII)}$$

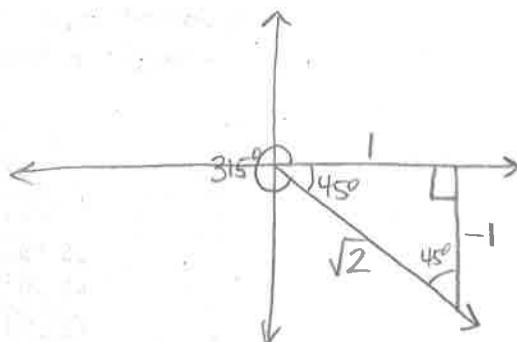
$$\sin \theta' = \sin 240^\circ$$

$$= \boxed{-\frac{\sqrt{3}}{2}}$$

* sin is \ominus in QIII

b) $\sec \frac{7\pi}{4}$

$$= \sec 315^\circ$$



$$\theta' = 45^\circ \text{ (in QIV)}$$

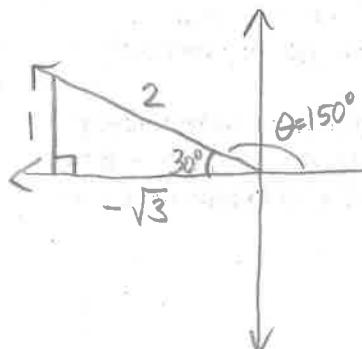
$$\sec \theta' = \sec \frac{7\pi}{4}$$

$$= \boxed{\sqrt{2}}$$

* sec is \oplus in QIV

c) $\tan \left(-\frac{19\pi}{6}\right) = \tan \frac{5\pi}{6} = \tan 150^\circ$

coterminal



$$\theta' = 30^\circ \text{ (in QII)}$$

$$\tan \theta' = \tan 150^\circ$$

$$= \frac{1}{-\sqrt{3}} = \boxed{-\frac{\sqrt{3}}{3}}$$

* tan is \ominus in QII

Quadrantal Angles

$\rightarrow 0^\circ, 90^\circ, 180^\circ, 270^\circ$ or $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$

- requires use of the UNIT CIRCLE (circle with $r = 1$).

* also, recall:

$$\sin \theta = \frac{y}{r}$$

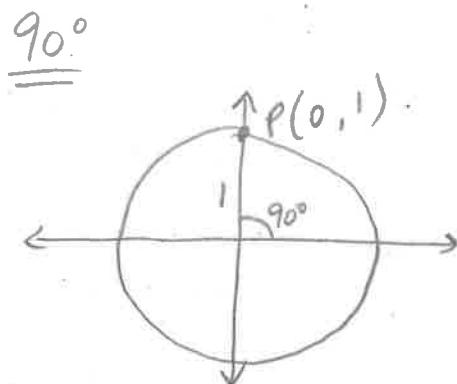
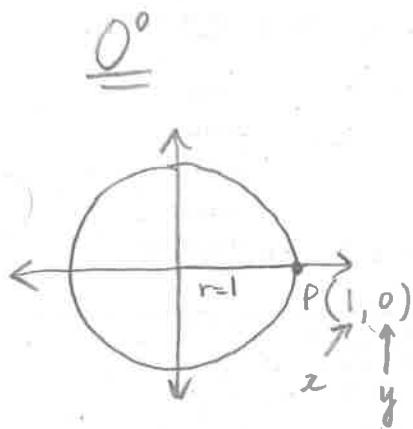
$$\cos \theta = \frac{x}{r}$$

$$\tan \theta = \frac{y}{x}$$

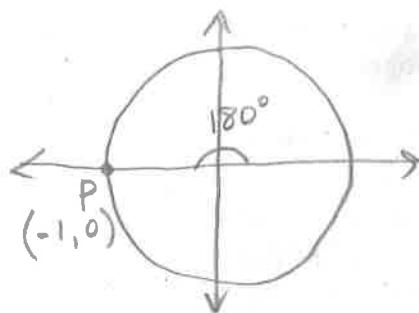
$$\csc \theta = \frac{r}{y}$$

$$\sec \theta = \frac{r}{x}$$

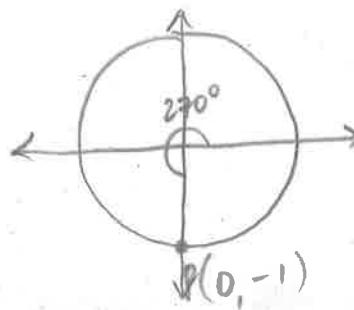
$$\cot \theta = \frac{x}{y}$$



180°



270°



eg3: Using the diagrams on the previous page,
find:

a) $\cos 0^\circ$

$$\cos 0^\circ = \frac{x}{r} = \frac{1}{1} = \boxed{1}$$

b) $\tan 90^\circ$

$$\tan 90^\circ = \frac{y}{x} = \frac{1}{0} = \boxed{\text{undefined}}$$

c) $\sin \pi$

$$\sin \pi = \sin 180^\circ = \frac{y}{r} = \frac{0}{1} = \boxed{0}$$

d) $\csc \frac{3\pi}{2}$

$$\csc \frac{3\pi}{2} = \csc 270^\circ$$

$$= \frac{r}{y} = \frac{1}{-1} = \boxed{-1}$$

e) $\sec(-2\pi)$

$$\sec -2\pi = \sec 0^\circ = \sec 0^\circ$$

$$= \frac{r}{x} = \frac{1}{1} = \boxed{1}$$

Using the same diagrams, fill in the following table:

θ	$\sin \theta$	$\cos \theta$	$\tan \theta$	$\csc \theta$	$\sec \theta$	$\cot \theta$
0° or 0	$\frac{0}{1} = 0$	$\frac{1}{1} = 1$	$\frac{0}{1} = 0$	$\frac{1}{0} = \text{undef.}$	$\frac{1}{1} = 1$	$\frac{1}{0} = \text{undef.}$
90° or $\frac{\pi}{2}$	$\frac{1}{1} = 1$	$\frac{0}{1} = 0$	$\frac{1}{0} = \text{undef.}$	$\frac{1}{1} = 1$	$\frac{1}{0} = \text{undef.}$	$\frac{0}{1} = 0$
180° or π	$\frac{0}{1} = 0$	$\frac{-1}{1} = -1$	$\frac{0}{-1} = 0$	$\frac{1}{0} = \text{undef.}$	$\frac{1}{-1} = -1$	$\frac{-1}{0} = \text{undef.}$
270° or $\frac{3\pi}{2}$	$\frac{-1}{1} = -1$	$\frac{0}{1} = 0$	$\frac{-1}{0} = \text{undef.}$	$\frac{1}{-1} = -1$	$\frac{1}{0} = \text{undef.}$	$\frac{0}{-1} = 0$
	$* \frac{y}{r}$	$* \frac{x}{r}$	$* \frac{y}{x}$	$* \frac{r}{y}$	$* \frac{r}{x}$	$* \frac{x}{y}$

eg4: Evaluate by showing ratio:

a) $\cos 540^\circ$

$$\cos 540^\circ = \cos 180^\circ$$

$$= \frac{x}{r} = \frac{-1}{1}$$

$\boxed{-1}$

b) $\cot \frac{9\pi}{2}$

$$\cot \frac{9\pi}{2} = \cot \frac{\pi}{2}$$

$$= \cot 90^\circ$$

$$= \frac{x}{y} = \frac{0}{1}$$

$\boxed{0}$

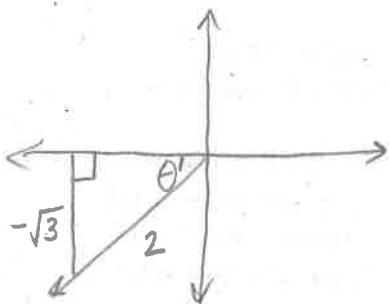
Finding θ

- to find θ for special or quadrantal angles,
reverse the reference angle or unit circle processes.

eg5: Find the smallest positive θ (in deg. and rad.)
such that:

a) $\sin \theta = -\frac{\sqrt{3}}{2}$

\sin first neg. in QIII

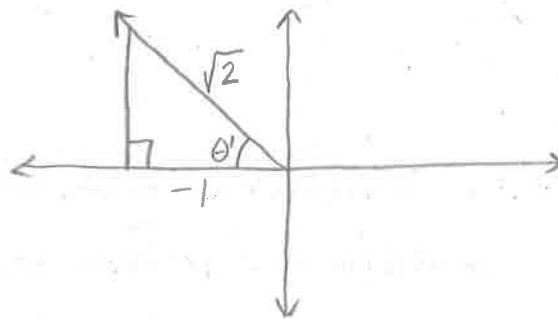


$$\theta' = 60^\circ$$

$$\theta = \boxed{240^\circ \text{ or } \frac{4\pi}{3}}$$

b) $\sec \theta = -\sqrt{2}$

\sec first neg. in QII



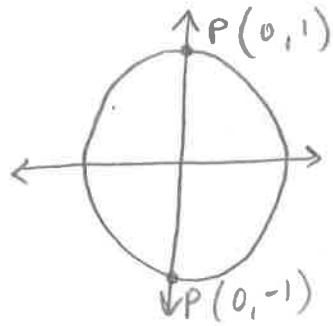
$$\theta' = 45^\circ$$

$$\theta = \boxed{135^\circ \text{ or } \frac{3\pi}{4}}$$

c) $\tan \theta = \text{undefined}$

$$\tan \theta = \frac{y}{x} = \frac{\pm 1}{0}$$

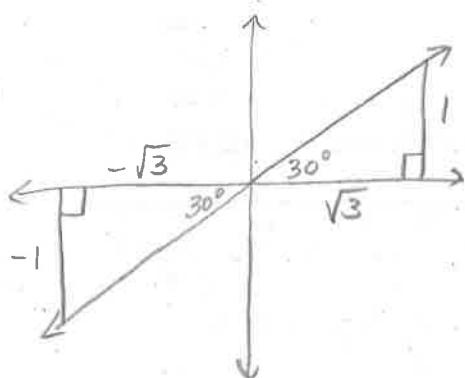
$$\theta = \boxed{90^\circ \text{ OR } \frac{\pi}{2}}$$



q6: Find exactly all θ , $0^\circ \leq \theta < 360^\circ$, such that:

a) $\tan \theta = \frac{1}{\sqrt{3}}$

* $\tan \theta$ in Qs I and III

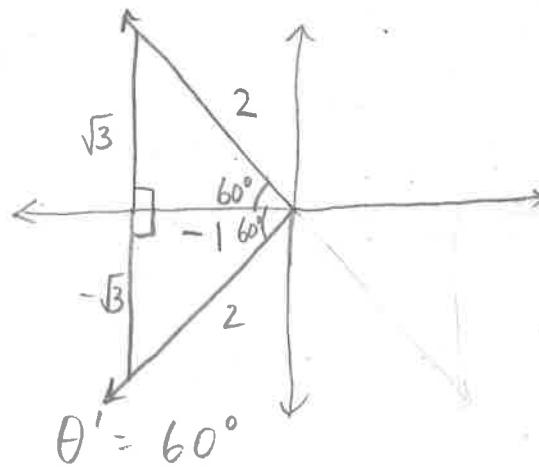


$$\theta' = 30^\circ$$

$$\begin{aligned} \text{QI: } \theta &= 30^\circ \\ \text{QIII: } \theta &= 210^\circ \end{aligned}$$

b) $\cos \theta = -\frac{1}{2}$

* $\cos \theta$ in Qs II and III



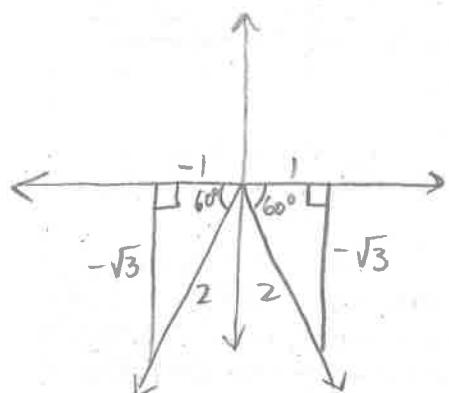
$$\theta' = 60^\circ$$

$$\begin{aligned} \text{QII: } \theta &= 120^\circ \\ \text{QIII: } \theta &= 240^\circ \end{aligned}$$

eg7: Find exactly all x , $0 \leq x < 2\pi$, such that:

a) $\csc x = \frac{-2}{\sqrt{3}}$

$\csc \theta$ in Qs III + IV



$$x' = 60^\circ = \frac{\pi}{3}$$

QIII:

$$x = \frac{4\pi}{3}$$

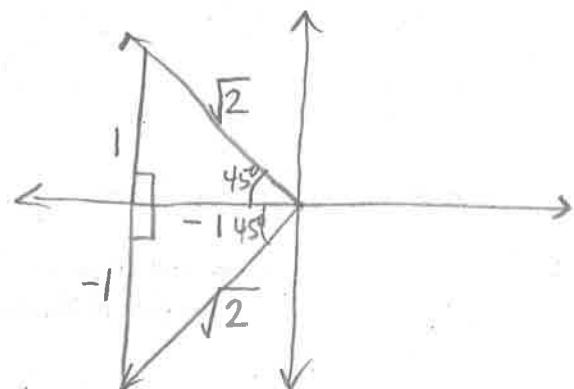
or

$$x = \frac{5\pi}{3}$$

QIV:

b) $\sec x = -\sqrt{2}$

$\sec \theta$ in Qs II and III



$$x' = 45^\circ = \frac{\pi}{4}$$

QII:

$$x = \frac{3\pi}{4}$$

or

$$x = \frac{5\pi}{4}$$

QIII:

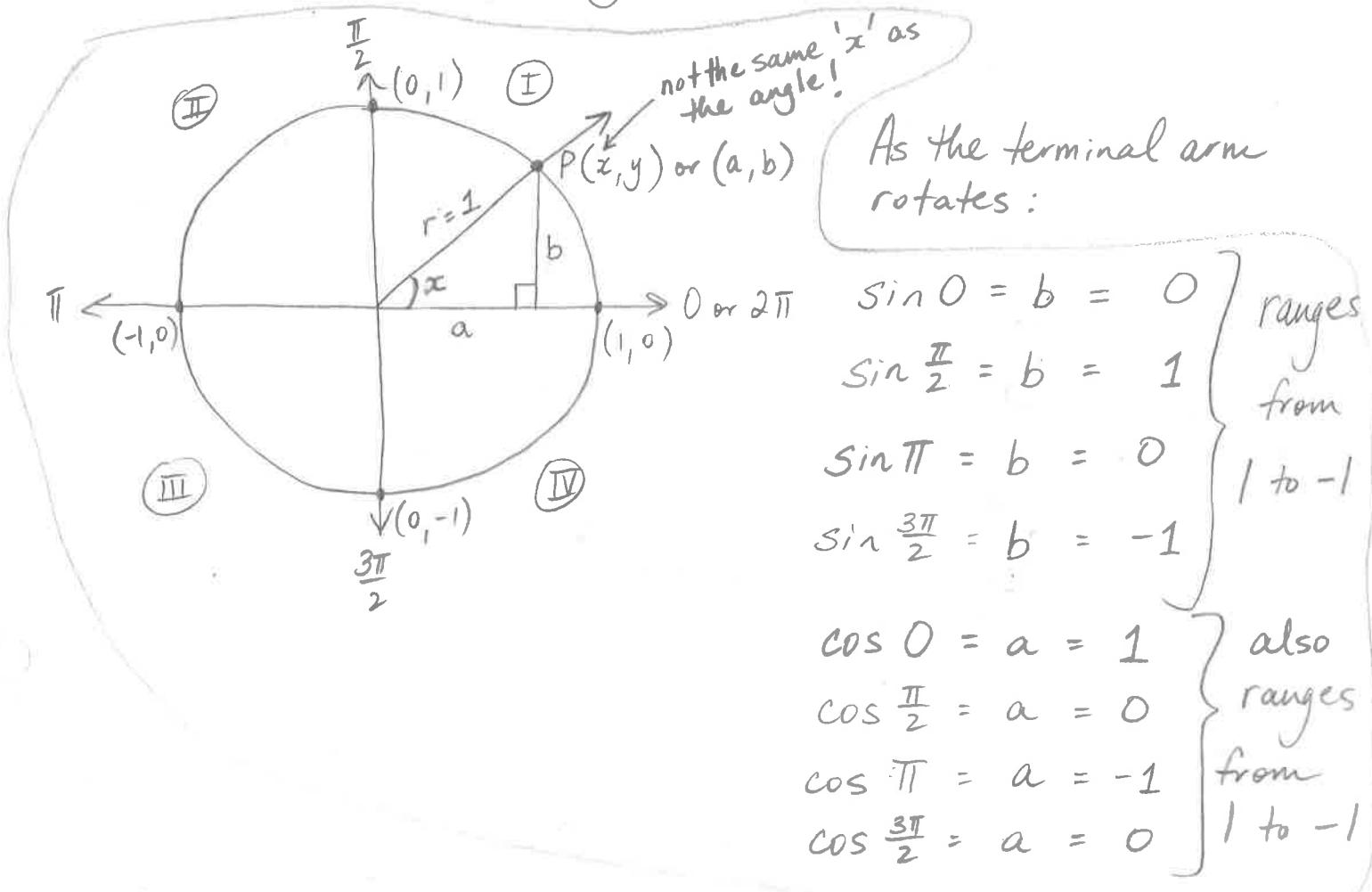
* glance at p. 272

p. 273

1-15

Ch. 6.4 - Graphing Basic Trig. Functions

- Consider, once again, a UNIT CIRCLE:



Consider the functions, then :

$$f(x) = \sin x \quad \text{and} \quad f(x) = \cos x$$

$$(y = \sin x) \qquad \qquad (y = \cos x)$$

As x varies from: (ie. as the terminal arm rotates from:)	$y = \sin x$ varies from:	$y = \cos x$ varies from:
0 to $\frac{\pi}{2}$	0 to 1	1 to 0
$\frac{\pi}{2}$ to π	1 to 0	0 to -1
π to $\frac{3\pi}{2}$	0 to -1	-1 to 0
$\frac{3\pi}{2}$ to 2π	-1 to 0	0 to 1

Also, note that :

$$\sin \frac{\pi}{6} = \frac{1}{2} = 0.5$$

$$\sin \frac{\pi}{4} = \frac{1}{\sqrt{2}} = 0.707$$

$$\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2} = 0.866$$

again, $\sin 0 = \underline{0}$ and $\sin \frac{\pi}{2} = \underline{1}$.

Therefore, $y = \sin x$ is NOT linear!

Further,

$$\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2} = 0.866$$

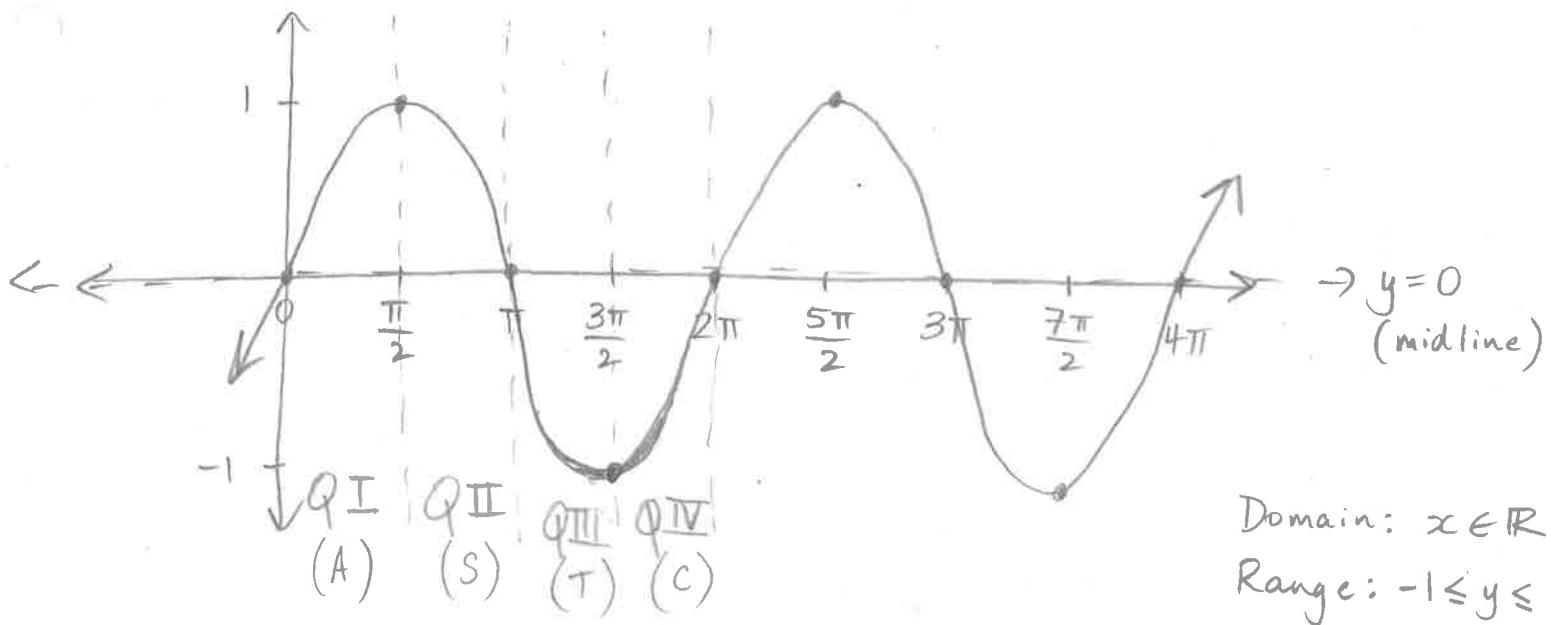
$$\cos \frac{\pi}{4} = \frac{1}{\sqrt{2}} = 0.707$$

$$\cos \frac{\pi}{3} = \frac{1}{2} = 0.5$$

again, $\cos 0 = \underline{1}$ and $\cos \frac{\pi}{2} = \underline{0}$.

Therefore, $y = \cos x$ is NOT linear!

Eg1: Graph $y = \sin x$ ($0 \leq x \leq 4\pi$)

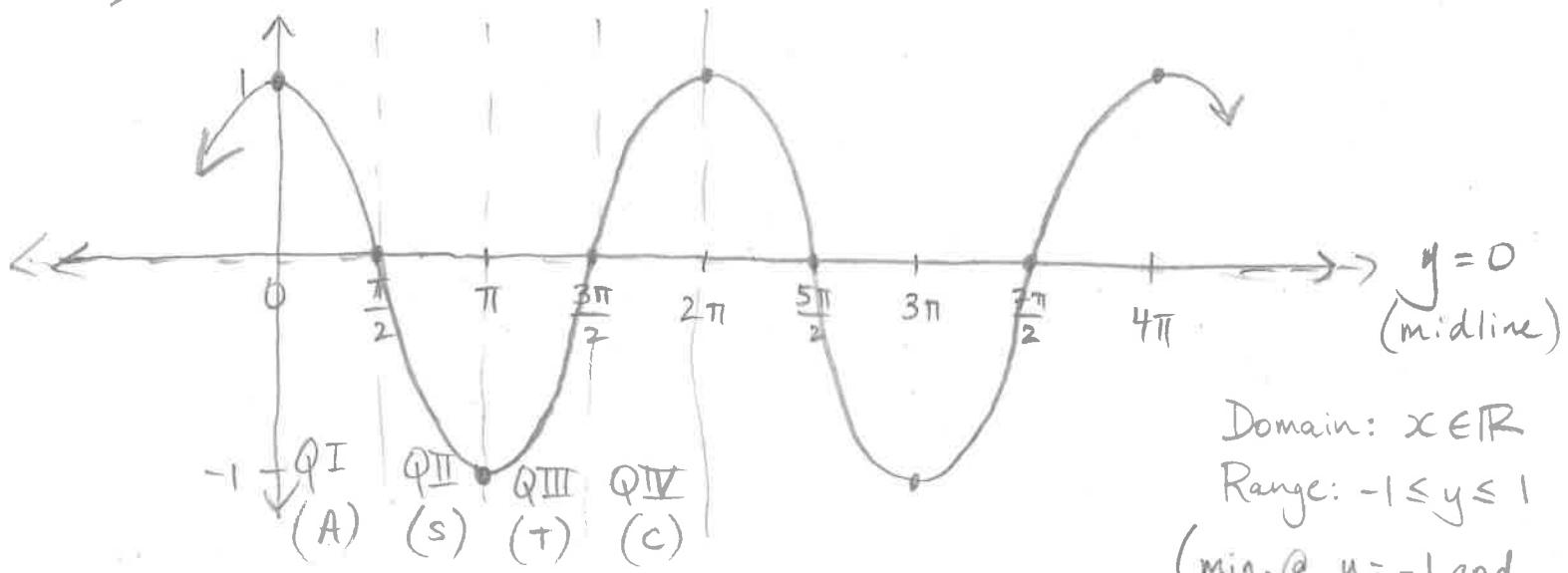


* note: this is the basic sine function.

It can be transformed:

$$\underline{y = a \sin(b(x-c)) + d}$$

Eg2: Graph $y = \cos x$ ($0 \leq x \leq 4\pi$)



* note: this is the basic cosine function.

It can be transformed:

$$\underline{y = a \cos(b(x-c)) + d}$$

Domain: $x \in \mathbb{R}$

Range: $-1 \leq y \leq 1$

(min. @ $y = -1$ and
max. @ $y = 1$)

- notice that these graphs repeat in successive intervals; they are PERIODIC in nature.
 - this, due to the existence of co-terminal angles.
- each repeated interval is called a PERIOD.

What is the period for: $y = \sin x$? 2π
 $y = \cos x$? 2π

In general: A function, f , is periodic if there exists a value c such that $f(x \pm c) = f(x)$ for all x in f 's domain.

Notice from graphs on previous page:

$$\sin x = \underline{\cos(x - \frac{\pi}{2})}$$

$$\sin x = \underline{\cos(x + \frac{3\pi}{2})}$$

$$\cos x = \underline{\sin(x + \frac{\pi}{2})}$$

Transformations of $y = \sin x$ and $y = \cos x$

General forms:

$$y = f(x) = a \sin(b(x-c)) + d$$

and

$$y = f(x) = a \cos(b(x-c)) + d$$

$\left. \begin{array}{l} a \neq 0 \\ b > 0 \\ (\text{in Math 12}) \end{array} \right\}$

* BEWARE! If given:

$$y = a \underset{\sin \text{ or } \cos}{\overset{\sin}{\cos}} (bx - c) + d,$$

you must factor out the b to find the true horizontal shift!

Terminology:

$$|a| = \text{AMPLITUDE}$$

Let m represent the minimum y -value and let M represent the maximum.
 $\text{Amplitude} = \frac{M-m}{2}$

$$\frac{2\pi}{|b|} = \frac{2\pi}{b} = \text{PERIOD}$$

(since $b > 0$)

Horizontal Translation: relies upon c ;
 (aka Phase Shift) if $c > 0$, shift Right
 if $c < 0$, shift Left

Vertical Translation: relies upon d ;
 (aka Vert. Displacement) if $d > 0$, shift Up

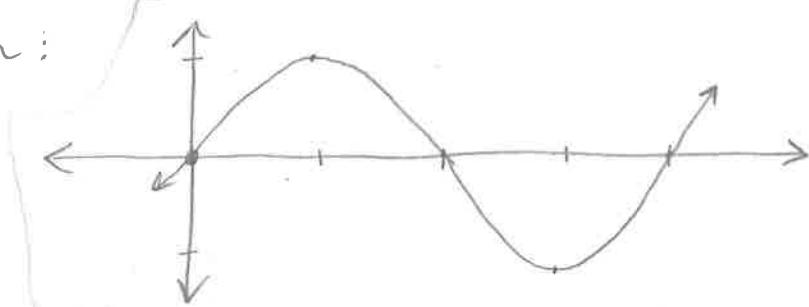
[*also: $y = d$ is the MIDLINe] if $d < 0$, shift Down.

Altering the Amplitude

- if 'a' is negative, reflect the function over the x-axis.

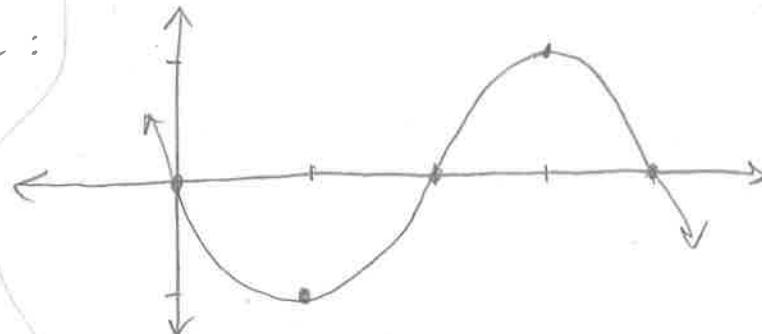
Thus, there are 4 graph themes:

① sin:



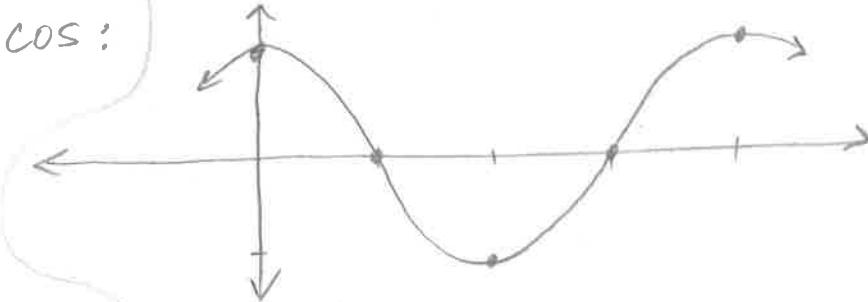
- starts on midline
- ends on midline
- UP first

② sin:



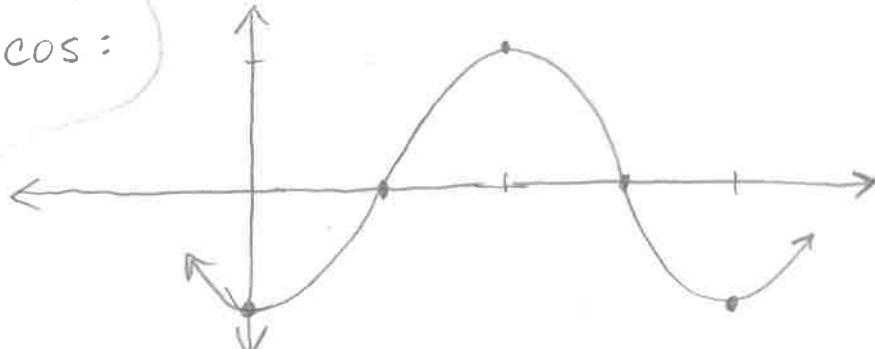
- starts on midline
- ends on midline
- DOWN first

③ cos:



- starts at max.
- ends at max.

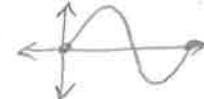
④ cos:



- starts at min.
- ends at min.

eg1: Graph one period of $y = 2 \sin x$.

$$a = 2 > 0 \rightarrow +\sin \text{ function}$$



$$\text{Amplitude} = |a| = |2| = 2$$

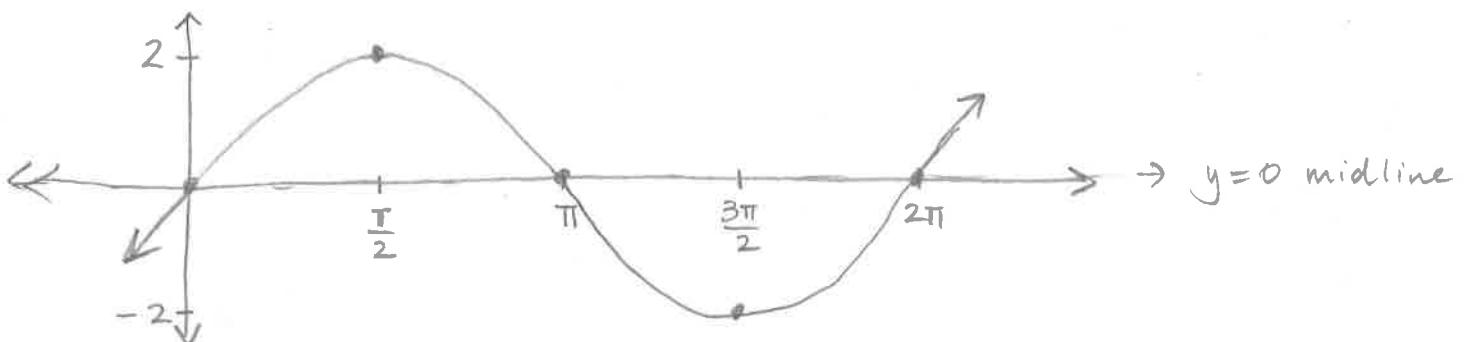
$$\text{Period} = \frac{2\pi}{|b|} = \frac{2\pi}{1} = 2\pi$$

Vert. displacement: $d = 0$ (none)

(so, midline @ $y = 0$)

Phase shift: $c = 0$ (no P.S.)

Start @ $(0, 0)$ End @ $(2\pi, 0)$



eg2: Graph one period of $y = -\frac{1}{2} \cos x$.

$$a = -\frac{1}{2} < 0 \rightarrow -\cos \text{ fcn}$$



$$\text{Amplitude} = |a| = \left| -\frac{1}{2} \right| = \frac{1}{2}$$

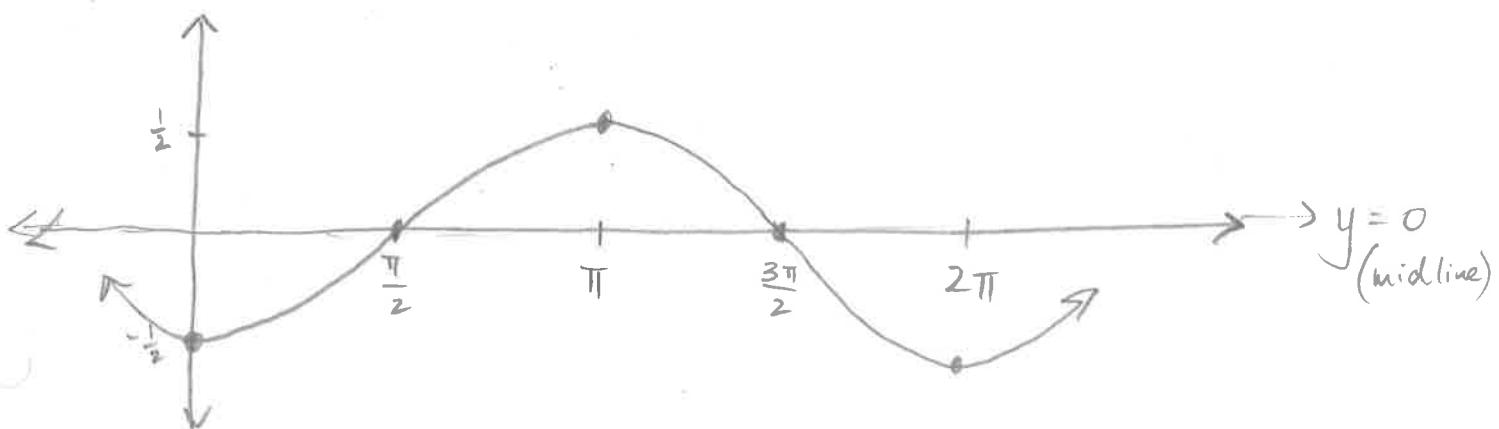
$$\text{Period} = \frac{2\pi}{|b|} = \frac{2\pi}{1} = 2\pi$$

Vert. displ.: $d = 0$ (none)

(so, midline @ $y = 0$)

Phase Shift: $c = 0$ (No P.S.)

Start @ $(0, -\frac{1}{2})$ End @ $(2\pi, -\frac{1}{2})$

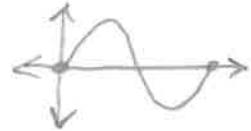


Altering the Period

- in Math 12, $b > 0$.
- remember, the period of sine and cosine functions equals $\frac{2\pi}{|b|}$.

eg 3: Graph one period of $y = \sin 2x$.

$$a = 1 > 0 \rightarrow \oplus \text{ sin function} \rightarrow$$



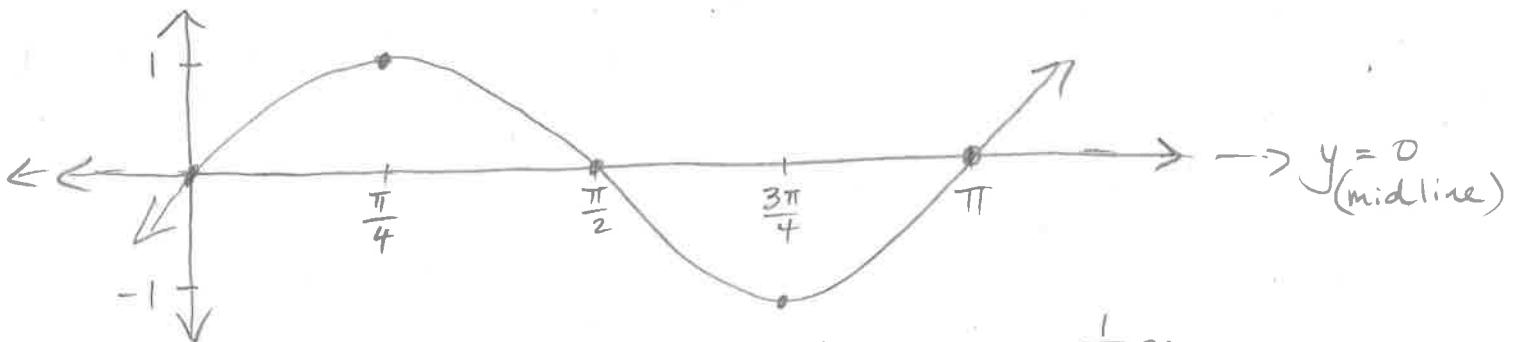
$$\text{Amplitude} = |a| = |1| = 1$$

$$\text{Period} = \frac{2\pi}{|b|} = \frac{2\pi}{2} = \pi$$

Vert. displ.: $d = 0$ (none)
(midline @ $y = 0$)

Phase Shift: $c = 0$ (none)

Start @ $(0, 0)$ End @ $(\pi, 0)$



eg 4: Graph one period of $y = \cos \frac{1}{3}x$.

$$a = 1 > 0 \rightarrow \oplus \text{ cos fcn.} \rightarrow$$



$$\text{Amplitude} = |a| = |1| = 1$$

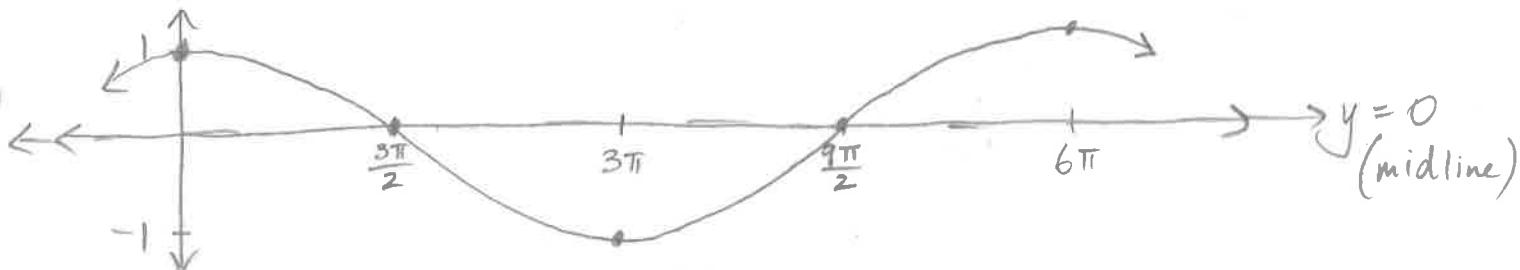
$$\text{Period} = \frac{2\pi}{|b|} = \frac{2\pi}{(\frac{1}{3})} = 6\pi$$

Vert. displ.: $d = 0$ (none)
(midline @ $y = 0$)

Phase Shift: $c = 0$ (none)

Start @ $(0, 1)$

End @ $(6\pi, 1)$



Altering the Vertical Displacement

- relies upon d :

i) if $d > 0$, shift UP.

ii) if $d < 0$, shift DOWN.

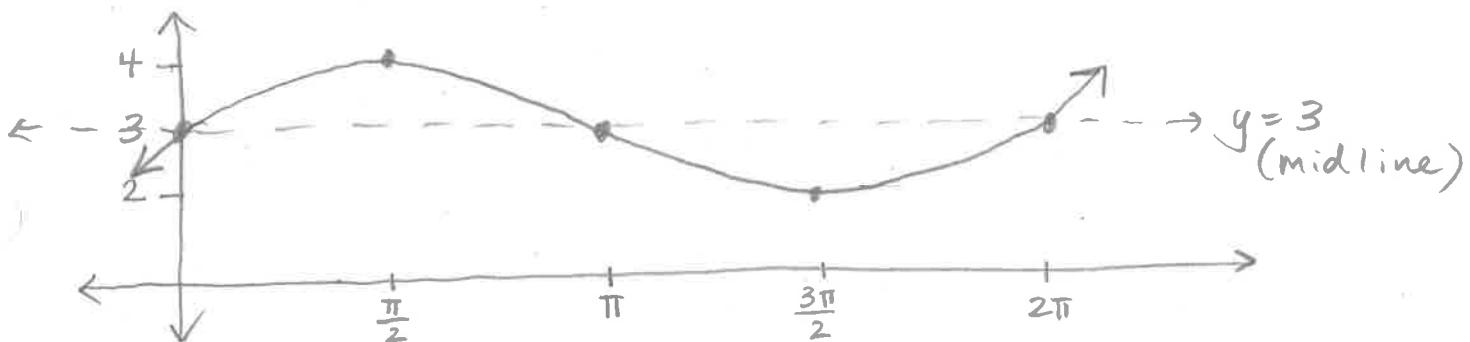
eg5: Graph one period of $y = \sin x + 3$.

$a = 1 > 0 \rightarrow +\sin$ fxn. \rightarrow 

$$\text{Amplitude} = |a| = |1| = 1 \quad \text{Period} = \frac{2\pi}{|b|} = \frac{2\pi}{1} = 2\pi$$

Vertical displ.: $d = 3 \rightarrow$ UP 3 Phase shift: $c = 0$ (none)
(midline @ $y = 3$)

Start @ $(0, 3)$ End @ $(2\pi, 3)$



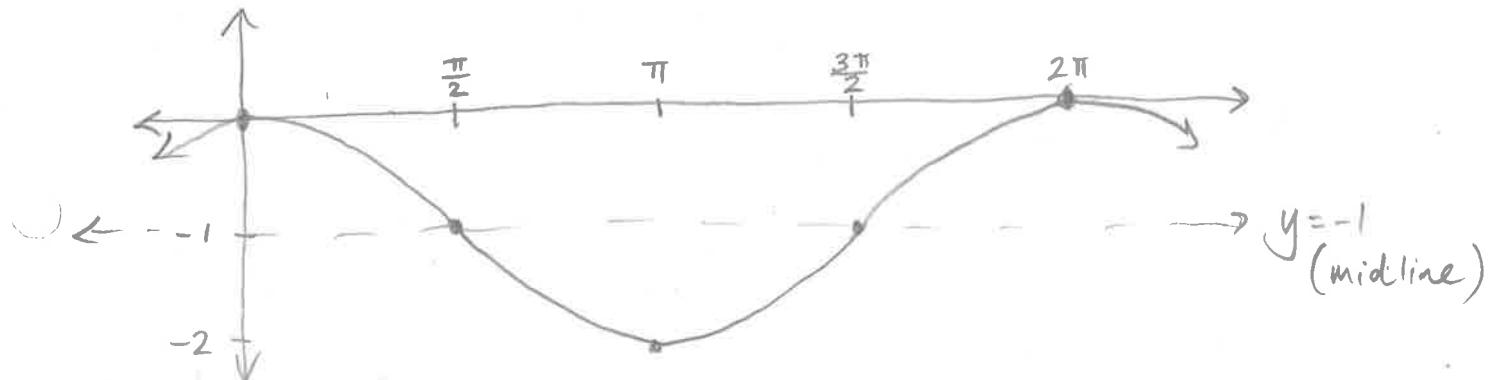
eg6: Graph one period of $y = \cos x - 1$.

$a = 1 > 0 \rightarrow +\cos$ fxn. \rightarrow 

$$\text{Amplitude} = |a| = |1| = 1 \quad \text{Period} = \frac{2\pi}{|b|} = \frac{2\pi}{1} = 2\pi$$

Vertical displ.: $d = -1 \rightarrow$ DOWN 1 Phase shift: $c = 0$ (none)
(midline @ $y = -1$)

Start @ $(0, 0)$ End @ $(2\pi, 0)$



Altering the Phase Shift

- relies upon c:

- i) if $c > 0$, shift RIGHT.
- ii) if $c < 0$, shift LEFT.

Eg 7: Graph one period of $y = \sin(x - \frac{\pi}{2})$.
 $a = 1 > 0 \rightarrow \oplus \sin \text{ fcn.} \rightarrow$

$$\text{Amplitude} = |a| = 1 \cdot 1 = 1$$

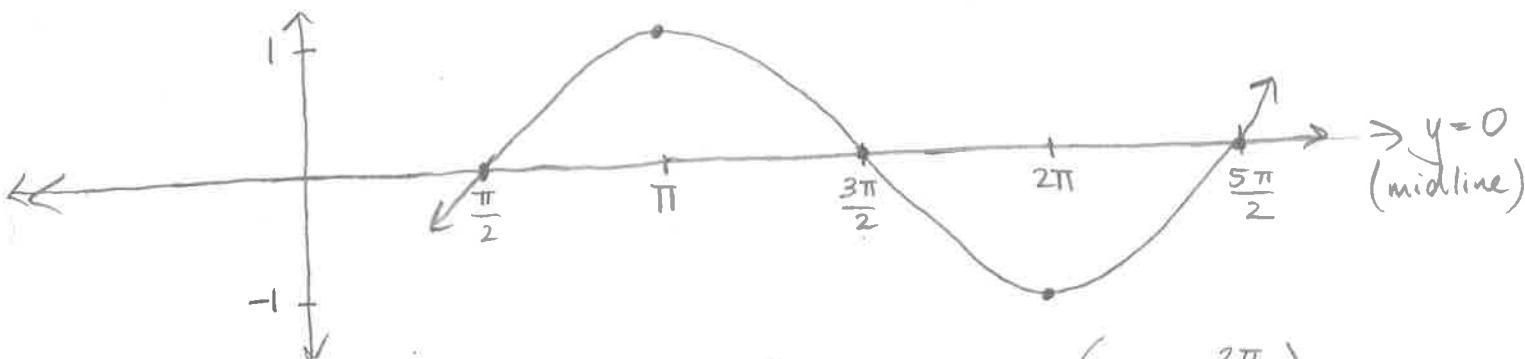
Vertical displ.: $d = 0$ (none)
 (midline @ $y = 0$)

Start @ $(\frac{\pi}{2}, 0)$

$$\text{Period} = \frac{2\pi}{|b|} = \frac{2\pi}{1} = 2\pi$$

Phase shift: $c = \frac{\pi}{2}$ ($\frac{\pi}{2}$ R)

End @ $(\frac{5\pi}{2}, 0)$



Eg 8: Graph one period of $y = \cos(x + \frac{2\pi}{3})$.
 $a = 1 > 0 \rightarrow \oplus \cos \text{ fcn.} \rightarrow$

$$\text{Amplitude} = |a| = 1 \cdot 1 = 1$$

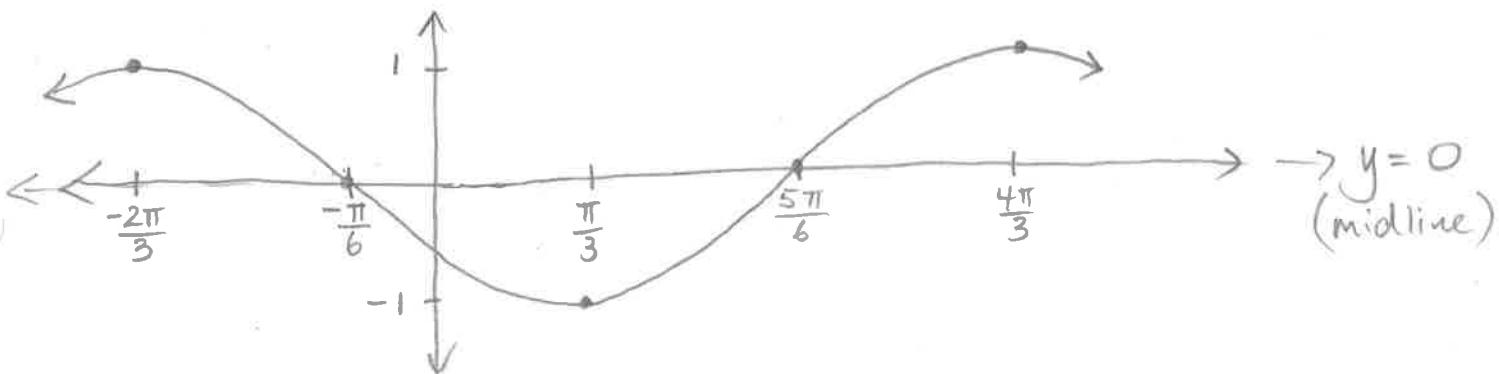
Vertical displ.: $d = 0$ (none)
 (midline @ $y = 0$)

Start @ $(-\frac{2\pi}{3}, 1)$

$$\text{Period} = \frac{2\pi}{|b|} = \frac{2\pi}{1} = 2\pi$$

Phase shift: $c = -\frac{2\pi}{3}$ ($\frac{2\pi}{3}$ L)

End @ $(\frac{4\pi}{3}, 1)$



Eg 9: Graph one period of each of the following:

a) $y = -3 \sin\left(\frac{3}{2}x + \frac{\pi}{2}\right) + 1$

$a = -3 < 1 \rightarrow \ominus \sin$ function \rightarrow

Amplitude = $|a| = |-3| = 3$ Period = $\frac{2\pi}{|b|} = \frac{2\pi}{\left(\frac{3}{2}\right)} = \frac{4\pi}{3}$

Vertical displ.: $d = 1$

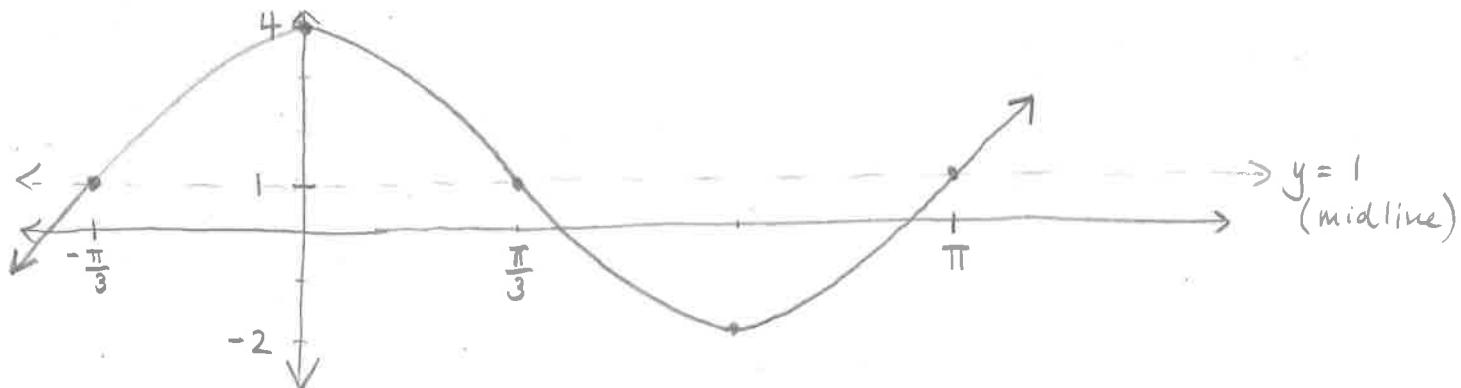
(midline @ $y = 1$)

Phase Shift: FACTOR out b !

$$y = -3 \sin\left(\frac{3}{2}(x + \frac{\pi}{3})\right) + 1$$

$$c = \frac{\pi}{3} (\frac{\pi}{3} \text{ L})$$

Start @ $(-\frac{\pi}{3}, 1)$ End @ $(\pi, 1)$



b) $y = 2 \cos\left(\frac{\pi}{6}(x - 2)\right) - 3$

$a = 2 > 1 \rightarrow \oplus \cos \rightarrow$

Amplitude = $|a| = |2| = 2$ Period = $\frac{2\pi}{|b|} = \frac{2\pi}{(\frac{\pi}{6})} = 12$

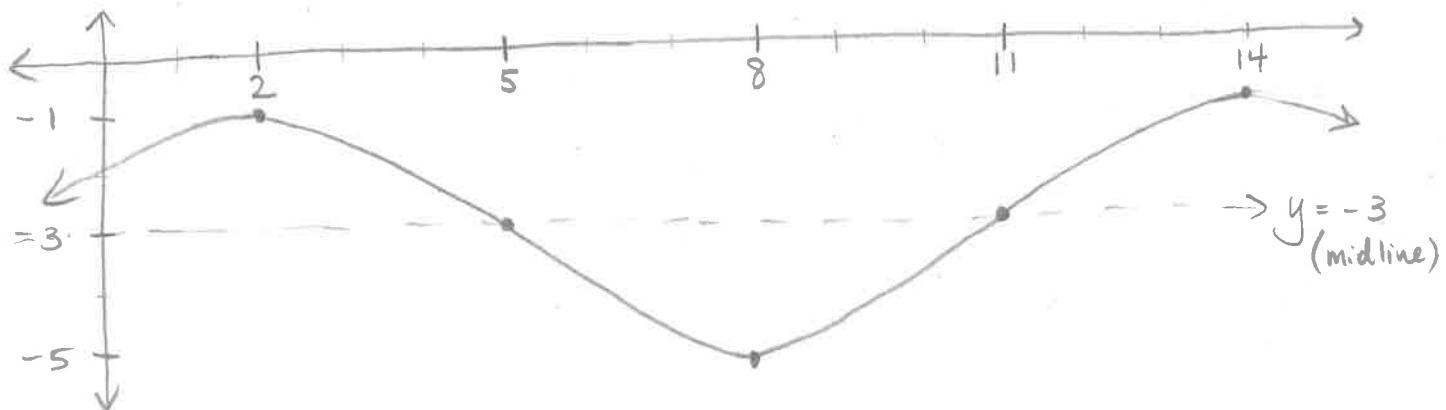
Vert. displ.: $d = -3$

(midline @ $y = -3$)

Phase Shift: $c = 2$ (2 R)

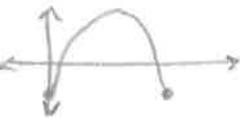
Start @ $(2, -1)$

End @ $(14, -1)$



$$c) y = -\cos \frac{\pi}{4}(x+3) + 1$$

$$a = -1 < 0 \rightarrow \ominus \cos \text{ fcn}$$



$$\text{Amplitude} = |a| = |-1| = 1$$

$$\text{Vert. displ. : } d = 1$$

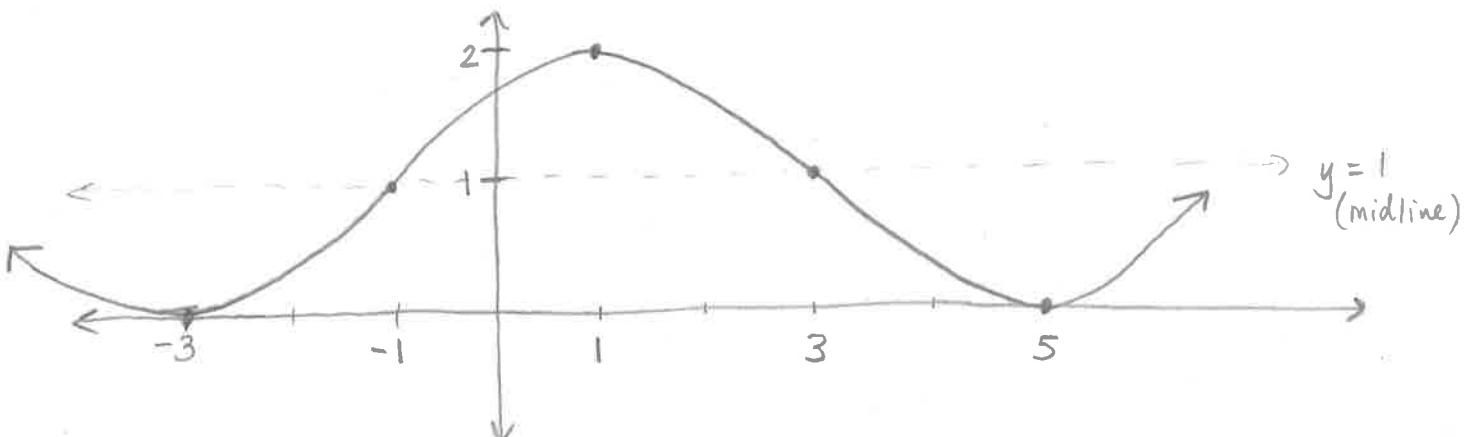
(midline @ $y=1$)

Start @ $(-3, 0)$

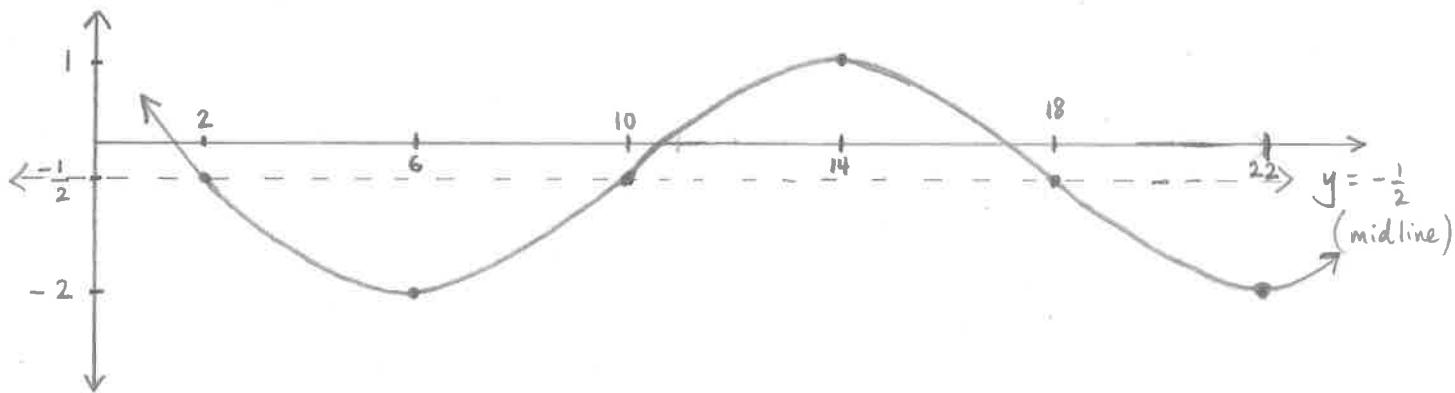
$$\text{Period} = \frac{2\pi}{|b|} = \frac{2\pi}{\left(\frac{\pi}{4}\right)} = 8$$

Phase Shift: $c = -3$ (3 L)

End @ $(5, 0)$



~~Ex 10:~~ Given the following graph, write an equation in terms of both sine and cosine.



$$\text{Amplitude} = \frac{3}{2} \rightarrow a = \pm \frac{3}{2}$$

$$\text{Period} = 16 \rightarrow \frac{2\pi}{b} = 16$$

$$\text{Midline @ } y = -\frac{1}{2}, d = -\frac{1}{2}$$

$$b = \frac{\pi}{8}$$

Phase Shift: $\ominus \sin \rightarrow c = 2$

$\ominus \cos \rightarrow c = 6$

$\oplus \sin \rightarrow c = 10$

$\oplus \cos \rightarrow c = 14$

$$y = -\frac{3}{2} \sin\left(\frac{\pi}{8}(x-2)\right) - \frac{1}{2}$$

$$y = -\frac{3}{2} \cos\left(\frac{\pi}{8}(x-6)\right) - \frac{1}{2}$$

$$y = \frac{3}{2} \sin\left(\frac{\pi}{8}(x-10)\right) - \frac{1}{2}$$

$$y = \frac{3}{2} \cos\left(\frac{\pi}{8}(x-14)\right) - \frac{1}{2}$$

Graphing $y = \tan x$

Recall, $y = \tan x$ is undefined when

$x = \underline{90^\circ \text{ or } \frac{\pi}{2}}$ and when $x = \underline{270^\circ \text{ or } \frac{3\pi}{2}}$,
along with all co-terminal angles to these.

These values are depicted on a tan graph
with vertical asymptotes.

Also recall, $y = \tan x = 0$ when

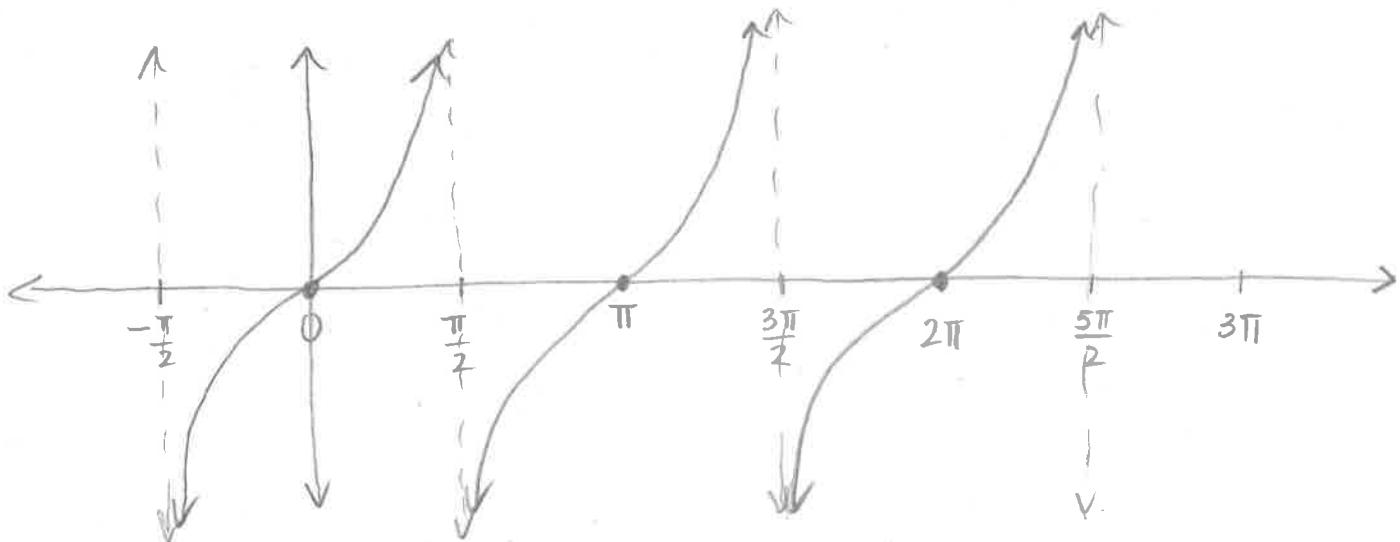
$x = \underline{0^\circ \text{ or } 0}$ and when $x = \underline{180^\circ \text{ or } \pi}$,
along with all co-terminal angles to these.

Of course, these values are depicted on a
tan graph as x -intercepts.

$y = \tan x$ is the 'basic' tan function:

$$\underline{y = 1 \tan(1(x - 0)) + 0}$$

eg1: Graph $y = \tan x$ (graph 3 periods)



Note: $\frac{\pi}{2} = 1.5708$ (rounded)

Period = $\frac{\pi}{2}$

$$\tan 1.57 = \frac{1256}{-109}$$

$$\tan 1.58 = -109$$

Domain: $x \neq \frac{\pi}{2} + n\pi$
 $(n \in \text{INTEGERS})$

Range: $y \in \mathbb{R}$.

$y = \tan x$ can be transformed:

$$y = a \tan(b(x-c)) + d$$

$$\text{Period} = \boxed{\frac{\pi}{|b|}}$$

Note: In Math 12,

$$\begin{aligned} a &= \pm 1 \\ d &= 0 \\ b &> 0 \end{aligned}$$

$$y = \underline{\pm \tan(b(x-c))}$$

- to calculate the first vertical asymptote, set $\underline{(b(x-c))} = \frac{\pi}{2}$. Add/subtract the period to find others.
- to calculate the first x -intercept, set $\underline{(b(x-c))} = 0$. Add/subtract the period to find others.

e.g 2: Graph 3 periods of each:

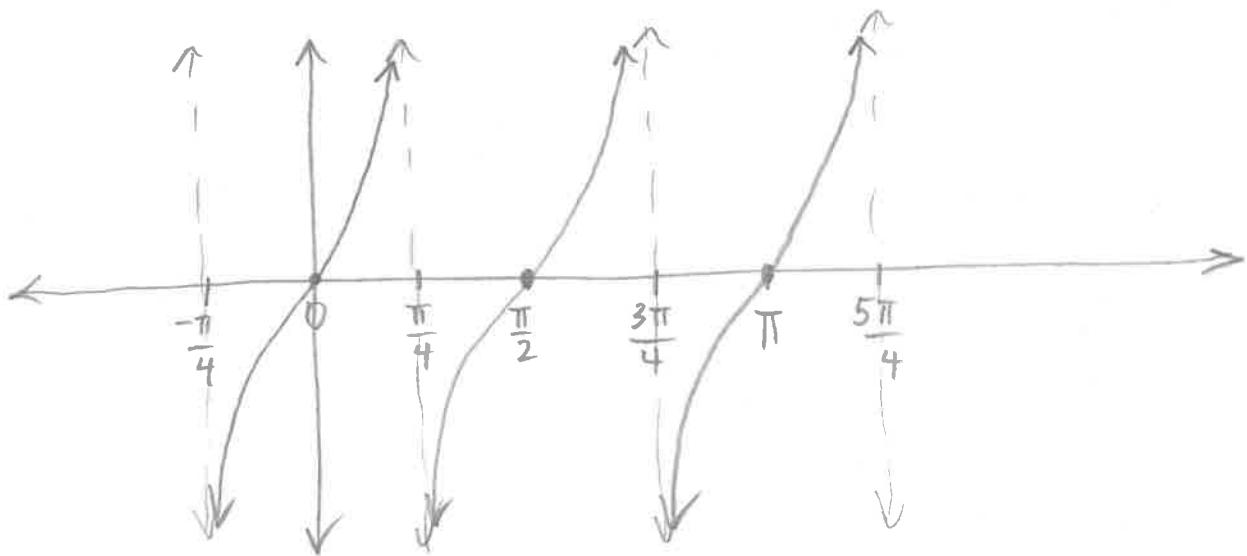
a) $y = \tan 2x$

$$2x = \frac{\pi}{2} \quad \text{Period} = \frac{\pi}{|b|} = \frac{\pi}{2}$$

$$x = \frac{\pi}{4}$$

$$2x = 0$$

$$x = 0$$



b) $y = -\tan \frac{1}{2}x$ * reflect over x -axis

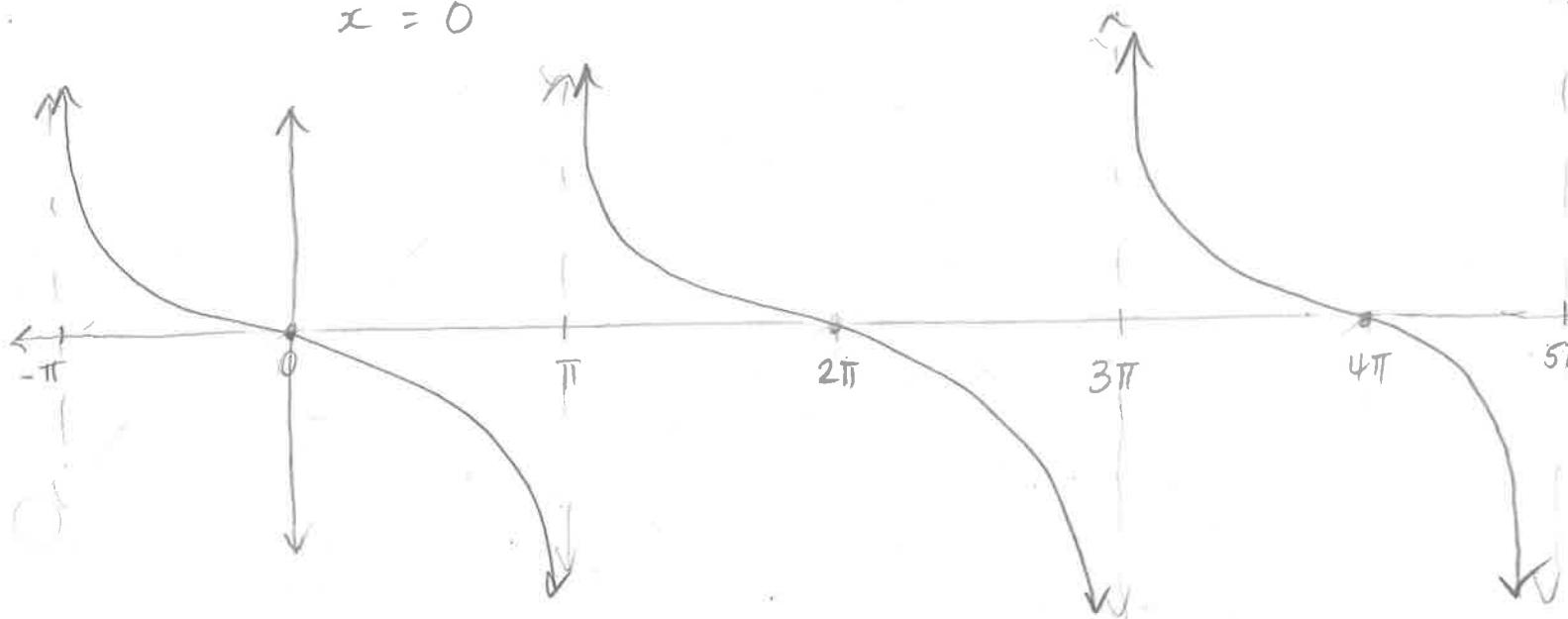
$$\frac{1}{2}x = \frac{\pi}{2}$$

$$x = \pi$$

$$\text{Period} = \frac{\pi}{|b|} = \frac{\pi}{\left(\frac{1}{2}\right)} = 2\pi$$

$$\frac{1}{2}x = 0$$

$$x = 0$$



c) $y = \tan\left(3\left(x - \frac{\pi}{6}\right)\right)$

$$3\left(x - \frac{\pi}{6}\right) = \frac{\pi}{2}$$

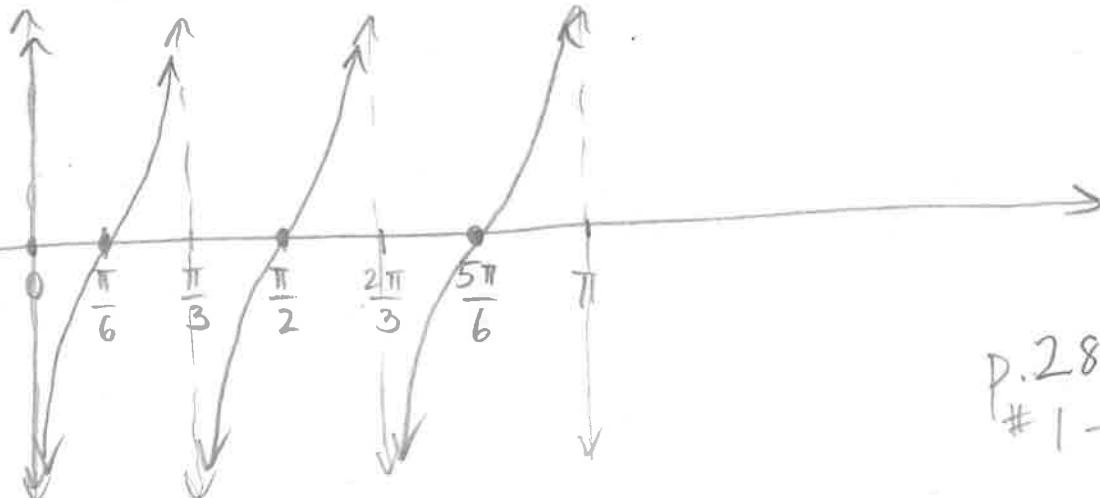
$$\text{Period} = \frac{\pi}{|b|} = \frac{\pi}{3}$$

$$x - \frac{\pi}{6} = \frac{\pi}{6}$$

$$3\left(x - \frac{\pi}{6}\right) = 0$$

$$x = \frac{\pi}{3}$$

$$x = \frac{\pi}{6}$$



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#1-10

Ch. 6.5 - Applications of Trig. Functions

- a motion that involves a pattern repeating itself over fixed time intervals is called harmonic motion.

↳ eg: pendulums, a Ferris Wheel, the amount of daylight during a year, tide patterns, etc.

Note: Period = $\frac{\# \text{ of units time}}{1 \text{ cycle}}$

Frequency = $\frac{\# \text{ of cycles}}{1 \text{ unit time}}$

Period = $\frac{1}{\text{freq.}}$

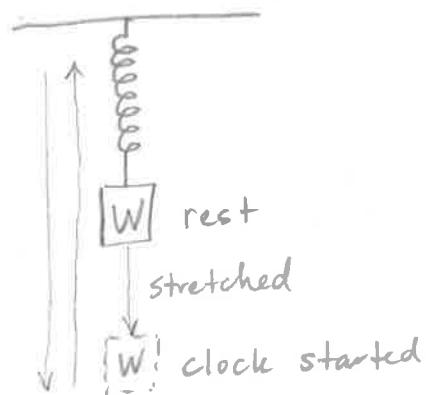
Frequency = $\frac{1}{\text{period}}$

eg: A car wheel takes $\frac{1}{4}$ s to turn around once. What is its frequency?

Period = $\frac{\frac{1}{4} \text{ s}}{1 \text{ cycle}}$

Frequency = $\frac{1}{\text{period}} = \frac{1}{(\frac{1}{4})} = \boxed{4 \text{ cycles/s}}$

e.g. 1: In a vacuum chamber, a weight is attached to a spring and set in motion by stretching the spring and then releasing it. The distance (in cm) that the spring is from its rest position at time t (in seconds) is given by: $d = -5 \cos 4\pi t$.

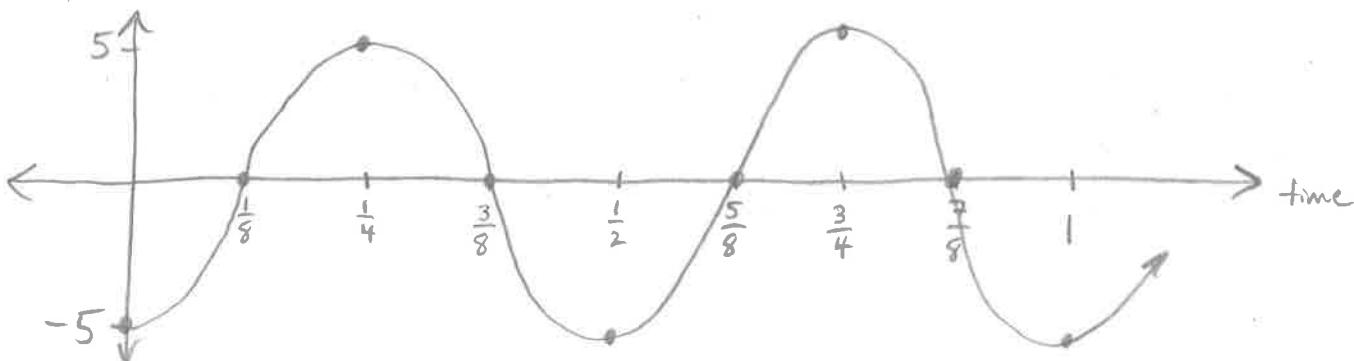


- a) How many cycles per second does the spring make?

$$\text{Period} = \frac{2\pi}{|b|} = \frac{2\pi}{4\pi} = \frac{1}{2} \rightarrow \frac{1}{2} \text{ s per cycle}$$

$$\text{Frequency} = \frac{1}{\text{period}} = \frac{1}{(\frac{1}{2})} = \boxed{2 \text{ cycles / s}}$$

- b) Graph 2 periods.



Eg 2: The voltage, E , of an electrical circuit has an amplitude of 220 volts and a frequency of 60 cycles per second. When $t = 0$, $E = 450$ volts (max. value). (t = time(s))
Write a periodic equation for this situation in terms of both sine and cosine.

$$\text{Frequency} = \frac{60 \text{ cycles}}{1 \text{ s}}$$

$$\text{Period} = \frac{1}{\text{freq.}} = \frac{1}{60} \rightarrow \frac{1}{60} \text{ s per cycle.}$$

$$\text{Period} = \frac{2\pi}{|b|} = \frac{2\pi}{b} = \frac{1}{60}$$

$$b = 120\pi$$

$$\text{Amplitude} = 220 \rightarrow a = \pm 220$$

$$\text{maximum} = 450 \rightarrow 450 - 220 = 230 \text{ (midline)}$$

$$d = 230$$

$$E = 220 \cos(120\pi t) + 230$$

$\frac{1}{4}$ of period

$$E = -220 \sin\left(120\pi\left(t - \frac{1}{240}\right)\right) + 230$$

$$E = -220 \cos\left(120\pi\left(t - \frac{1}{120}\right)\right) + 230$$

$\frac{1}{2}$ of period

$$E = 220 \sin\left(120\left(t - \frac{1}{80}\right)\right) + 230$$

$\frac{3}{4}$ of period

eg3: The monthly sales (S) of a seasonal product are approximated by:

$$S = 480 \cos \frac{\pi}{6}t + 760.$$

Graph the function and using the graph, find the months where 1000 units of sales were made. (t = time in months $\rightarrow t=1$ (January))

$$a = 480 > 0 \rightarrow \textcircled{4} \cos \text{fun} \rightarrow$$



$$\text{Amplitude} = |a| = |480| = 480$$

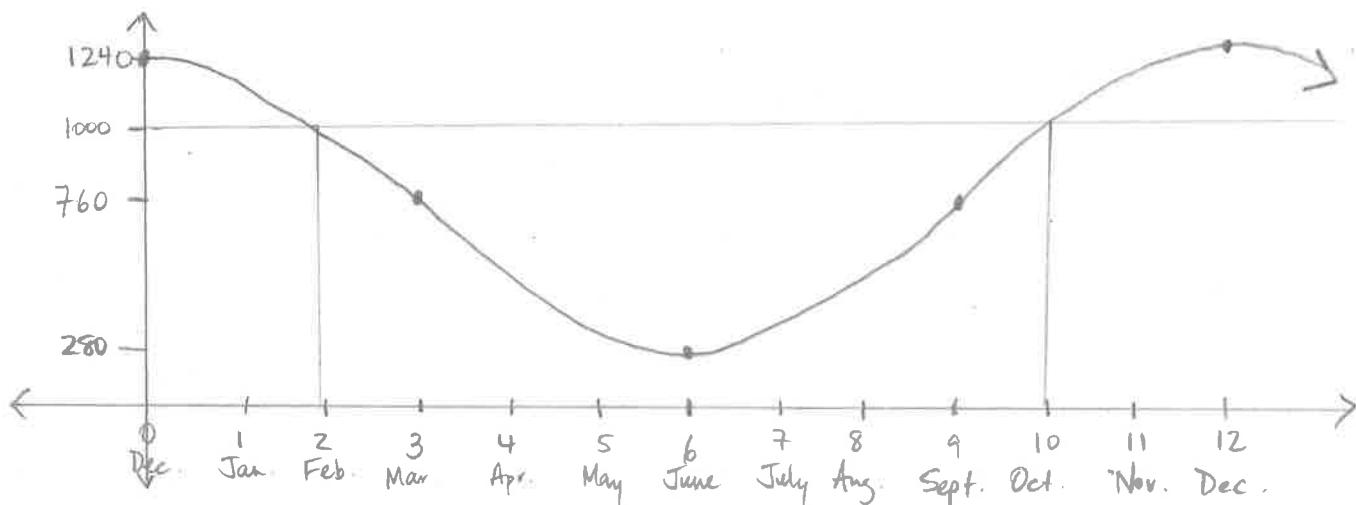
$$\text{Period} = \frac{2\pi}{|b|} = \frac{2\pi}{\left(\frac{\pi}{6}\right)} = 12$$

$$\text{Vert. displ. : } d = 760$$

(midline @ $y = 760$)

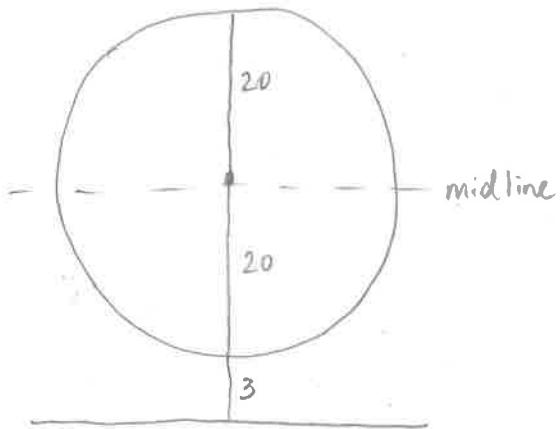
Phase Shift: none

Start @ $(0, 1240)$ End @ $(12, 1240)$



1000 units in February and October!

eg4: A Ferris wheel has a radius of 20m and rotates once every 60s. A rider enters the seat at the lowest point, 3m above the ground, and the clock is started. Graph the function and write an equation describing the function. Then, calculate the amount of time the rider spends above 30m.



let x = time (s)

let y = height (m)

$$\text{Amplitude} = 20 \Rightarrow a = \pm 20$$

$$\text{midline} @ y = 23$$

$$\text{Period} = 60$$

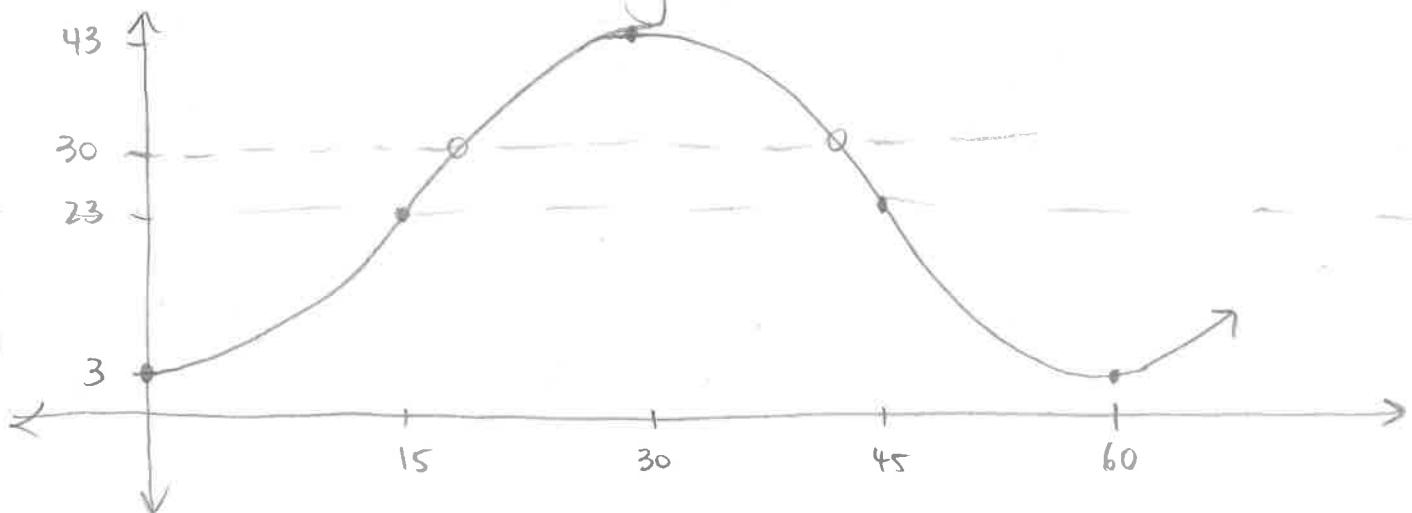
$$\frac{2\pi}{b} = 60 \rightarrow b = \frac{\pi}{30}$$

$$y = -20 \cos \frac{\pi}{30} x + 23$$

$$y = 20 \sin \left(\frac{\pi}{30} (x-15) \right) + 23$$

$$y = -20 \cos \left(\frac{\pi}{30} (x-30) \right) + 23$$

$$y = 20 \sin \left(\frac{\pi}{30} (x-45) \right) + 23$$



let $y = 30$

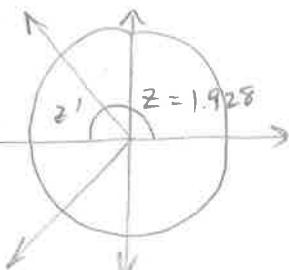
$$30 = -20 \cos \frac{\pi}{30} x + 23$$

$$\frac{-7}{20} = \cos \frac{\pi}{30} x$$

$$\frac{-7}{20} = \cos z$$

$$z = 1.928 \text{ (QII)}$$

$$z' = \pi - 1.928 \\ = 1.213$$



$$z = \pi + 1.213$$

$$= 4.355 \text{ (QIII)}$$

$$x = \frac{1.928}{\left(\frac{\pi}{30}\right)}, \quad \frac{4.355}{\left(\frac{\pi}{30}\right)}$$

$$x = 18.411, 41.587$$

$$\text{time above } 30 \text{ m} = 41.587 - 18.411$$

$$= 23.176 \text{ s}$$

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Ch. Review (p. 292)