

## Ch. 7.3 - Trigonometric Equations

When solving trig. equations, there are two types of solutions:

- i) Conditional solutions  $\rightarrow$  usually  $0 \leq x < 2\pi$
- ii) General form solutions  $\rightarrow$  eg: often  $x + 2n\pi$   
 $(n \in \text{INTEGERS})$

- there are infinitely many general form solutions if a conditional solution exists.

eg 1: Solve each of the following in two ways:

i)  $0 \leq x < 2\pi$

ii) General form

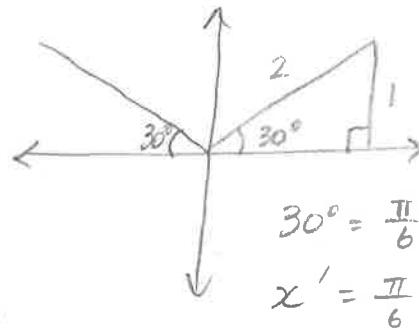
a)  $2 \sin x - 1 = 0$

i)  $2 \sin x = 1$

$$\sin x = \frac{1}{2}$$

$$x = \sin^{-1}\left(\frac{1}{2}\right)$$

$$\boxed{x = \frac{\pi}{6}, \frac{5\pi}{6}}$$



$$30^\circ = \frac{\pi}{6}$$

$$x' = \frac{\pi}{6}$$

ii) Any other angle coterminal to  $x$ :

solutions:  $\frac{\pi}{6} + 2n\pi, \frac{5\pi}{6} + 2n\pi$  ( $n$  an integer)

$$b) \cos x + \sqrt{2} = -\cos x$$

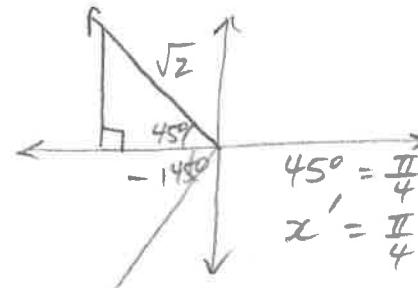
$$i) 2 \cos x = -\sqrt{2}$$

$$\cos x = -\frac{\sqrt{2}}{2} \quad (\text{'unrationalize'})$$

$$\cos x = -\frac{1}{\sqrt{2}}$$

$$x = \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$$

$$\boxed{x = \frac{3\pi}{4}, \frac{5\pi}{4}}$$



$$ii) \text{solutions: } \frac{3\pi}{4} + 2n\pi, \frac{5\pi}{4} + 2n\pi \quad (n \in \text{INTEGERS})$$

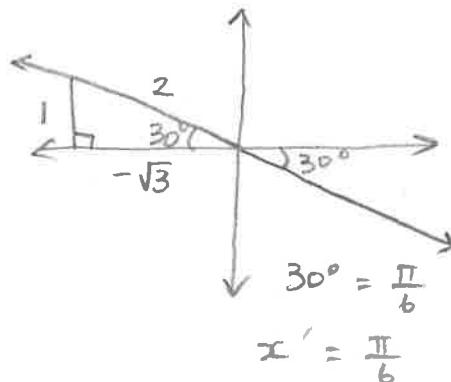
$$c) \sqrt{3} \tan x + 1 = 0$$

$$i) \sqrt{3} \tan x - 1$$

$$\tan x = -\frac{1}{\sqrt{3}}$$

$$x = \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$$

$$\boxed{x = \frac{5\pi}{6}, \frac{11\pi}{6}}$$



$$ii) \text{solutions: } \frac{5\pi}{6} + 2n\pi, \frac{11\pi}{6} + 2n\pi \quad (n \in \text{INTEGERS})$$

OR

$$\boxed{\frac{5\pi}{6} + n\pi}$$

$$d) \sin x \tan x = 2 \tan x$$

$$i) \sin x \tan x - 2 \tan x = 0$$

$$\tan x (\sin x - 2) = 0$$

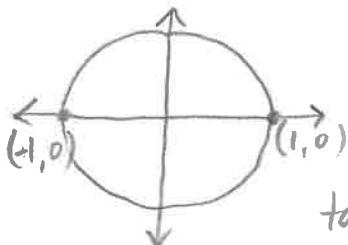
$$\tan x = 0 ; \sin x - 2 = 0$$

$$x = \tan^{-1} 0$$

$$\sin x = 2$$

NO SOLUTION

$$(-1 < \sin x < 1)$$



$$\tan = \frac{y}{x}$$

$$x = 0^\circ, 180^\circ$$

$$x = 0, \pi$$

$$ii) 0 + 2n\pi, \pi + 2n\pi$$

$n \in \text{INTEGERS}$

more simple:  $\boxed{n\pi}$

$$e) \sec^2 x - \sec x - 2 = 0$$

$$i) (\sec x - 2)(\sec x + 1) = 0$$

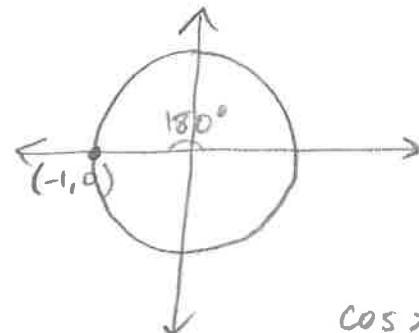
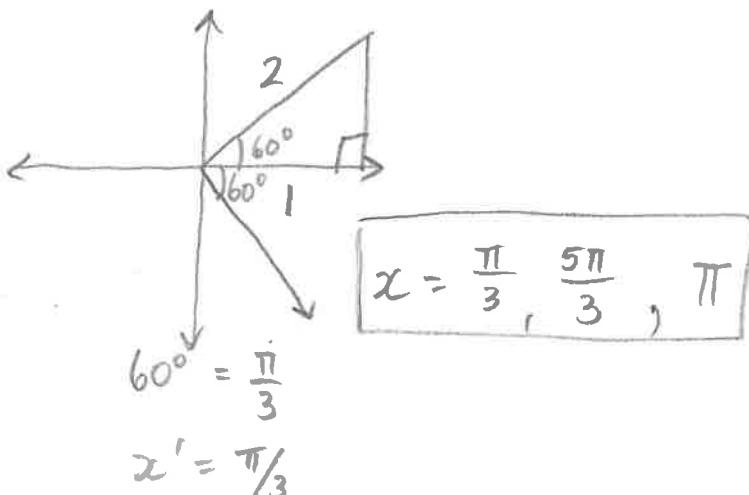
$$\sec x - 2 = 0 ; \sec x + 1 = 0$$

$$\sec x = 2$$

$$\sec x = -1$$

$$\cos x = \frac{1}{2}$$

$$\cos x = -1$$



$$\cos x = x\text{-value}$$

$$x = 180^\circ = \pi$$

$$iii) \frac{\pi}{3} + 2n\pi, \pi + 2n\pi, \frac{5\pi}{3} + 2n\pi$$

$n \in \text{INTEGERS}$

\* acceptable, but can be simpler: (answers are  $\frac{2\pi}{3}$  apart)

$$\frac{\pi}{3} + \frac{2n\pi}{3}$$

$$f) 2\cos^2 x + 3\sin x - 3 = 0$$

$$i) 2(1-\sin^2 x) + 3\sin x - 3 = 0$$

$$2 - 2\sin^2 x + 3\sin x - 3 = 0$$

$$-2\sin^2 x + 3\sin x - 1 = 0$$

$$2\sin^2 x - 3\sin x + 1 = 0$$

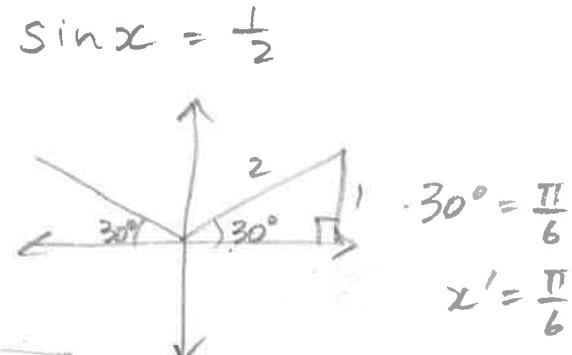
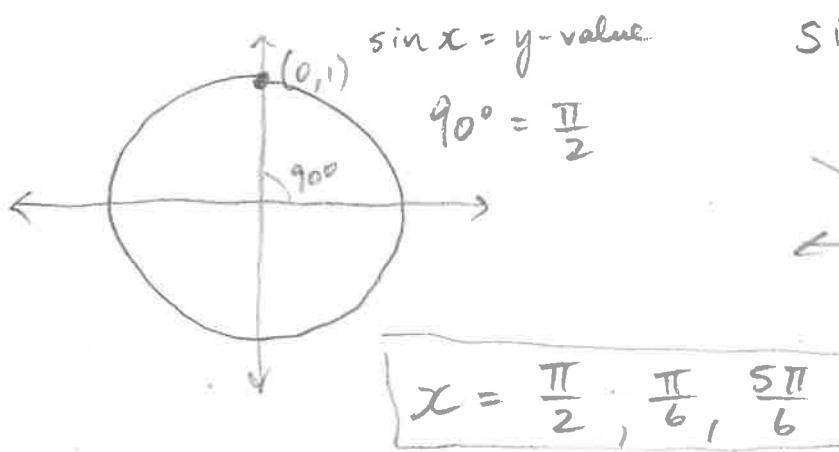
$$2\sin^2 x - 2\sin x - 1\sin x + 1 = 0$$

$$2\sin x (\sin x - 1) - 1(\sin x - 1) = 0$$

$$(\sin x - 1)(2\sin x - 1) = 0$$

$$\sin x - 1 = 0 \quad ; \quad 2\sin x - 1 = 0$$

$$\sin x = 1 \quad ; \quad 2\sin x = 1$$



$$ii) \frac{\pi}{6} + 2n\pi, \frac{\pi}{2} + 2n\pi, \frac{5\pi}{6} + 2n\pi \quad (n \in \text{INTEGERS})$$

no simplified solution possible.

$$g) \cos x + 1 = \sin x$$

$$i) (\cos x + 1)^2 = \sin^2 x \quad * \text{ could introduce extraneous solution(s)} \Rightarrow \text{CHECK!}$$

$$\cos^2 x + 2\cos x + 1 = \sin^2 x$$

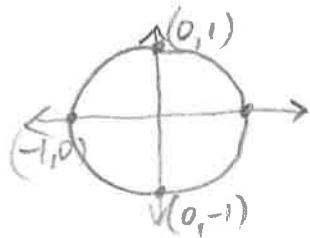
$$\cos^2 x + 2\cos x + 1 = 1 - \cos^2 x$$

$$2\cos^2 x + 2\cos x = 0$$

$$2\cos x (\cos x + 1) = 0$$

$$2\cos x = 0 \quad ; \quad \cos x + 1 = 0$$

$$\cos x = 0 \quad ; \quad \cos x = -1$$



$\cos x = x\text{-value}$

$$x = \frac{\pi}{2}, \pi, \frac{3\pi}{2}$$

Check!

$$ii) \frac{\pi}{2} + 2n\pi,$$

$$\pi + 2n\pi$$

$(n \in \text{INTEGERS})$

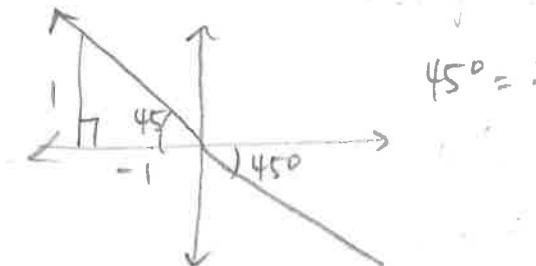
$$x = \frac{\pi}{2}, \pi$$

$$h) 4 \tan \frac{x}{2} + 4 = 0$$

$$i) 4 \tan \frac{x}{2} = -4$$

$$\tan \frac{x}{2} = -1$$

$$\frac{x}{2} = \tan^{-1}(-1)$$



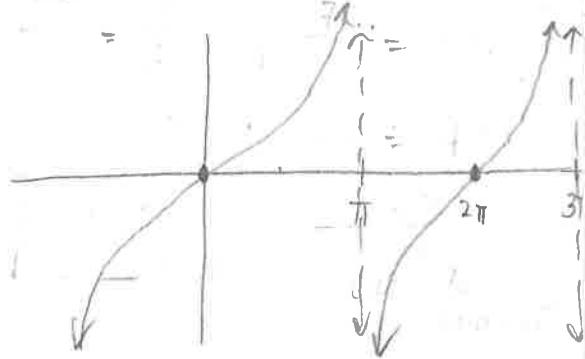
$$45^\circ = \frac{\pi}{4}$$

$$\frac{x}{2} = \frac{3\pi}{4} \quad ; \quad \frac{x}{2} = \frac{7\pi}{4}$$

$$x = \frac{3\pi}{2}, \frac{7\pi}{2} \quad (\text{outside domain})$$

$$ii) \frac{3\pi}{2} + 2n\pi$$

$(n \in \text{INTEGERS})$



$$i) \csc^2 x - 2 \cot x - 4 = 0$$

$$i) 1 + \cot^2 x - 2 \cot x - 4 = 0$$

$$\cot^2 x - 2 \cot x - 3 = 0$$

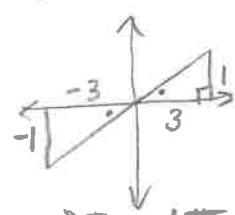
$$(\cot x - 3)(\cot x + 1) = 0$$

$$\cot x = 3$$

$$\cot x = -1$$

$$\tan x = \frac{1}{3}$$

$$\tan x = -1$$



non-special ratio  
(need calc.)

$$x = \tan^{-1}\left(\frac{1}{3}\right)$$

$$x = 0.3218, 0.3218 + \pi$$

$$x = 0.3218, \underbrace{3.4633}_{\pi \text{ apart}}, \underbrace{\frac{3\pi}{4}}_{\pi \text{ apart}}, \underbrace{\frac{7\pi}{4}}_{\pi \text{ apart}}$$

$$ii) 0.3218 + n\pi, \frac{3\pi}{4} + n\pi \quad (n \in \text{INTEGERS})$$

$$j) \cos^2 x - 3 \cos x - 2 = 0$$

$$i) \cos x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\cos x = \frac{3 \pm \sqrt{(-3)^2 - 4(1)(-2)}}{2(1)}$$

$$\cos x = \frac{3 \pm \sqrt{17}}{2}$$

$$\cos x = 3.5616, -0.5616$$

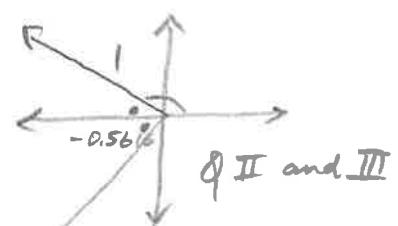
reject  
 $(-1 \leq \cos x \leq 1)$

$$ii) 2.1671 + 2n\pi, 4.1161 + 2n\pi$$

$$\text{see h: } x = \frac{3\pi}{4}, \frac{7\pi}{4}$$

$$\cos x = -0.5616$$

$$x = \cos^{-1}(-0.5616)$$



$$x = 2.1671 \quad (\text{QII})$$

$$x' = \pi - 2.1671$$

$$x' = 0.9745$$

$$\frac{x = \pi + x'}{x = 4.1161} \quad (\text{QIII})$$

$$k) \quad 6\sin^2 2x - \sin 2x - 1 = 0$$

$$i) \quad 6\sin^2 2x - 3\sin 2x + 2\sin 2x - 1 = 0$$

$$3\sin 2x(2\sin 2x - 1) + 1(2\sin 2x - 1) = 0$$

$$(2\sin 2x - 1)(3\sin 2x + 1) = 0$$

$$2\sin 2x = 1 \quad ; \quad 3\sin 2x = -1$$

$$\sin 2x = \frac{1}{2} \quad ; \quad \sin 2x = -\frac{1}{3}$$

special

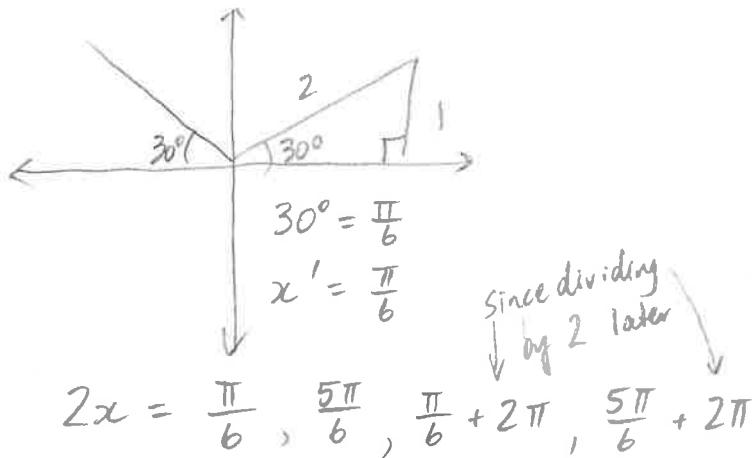
non-special

$$2x = \sin^{-1}\left(\frac{1}{2}\right)$$

Quad I, II

$$2x = \sin^{-1}\left(-\frac{1}{3}\right)$$

Quad III, IV



$$x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$$

$\pi$  apart  $\pi$  apart

$$x' = -0.3398$$

$$\text{Quad III: } \pi + 0.3398 \\ = 3.481$$

$$\text{Quad IV: } 2\pi - 0.3398 \\ = 5.943$$

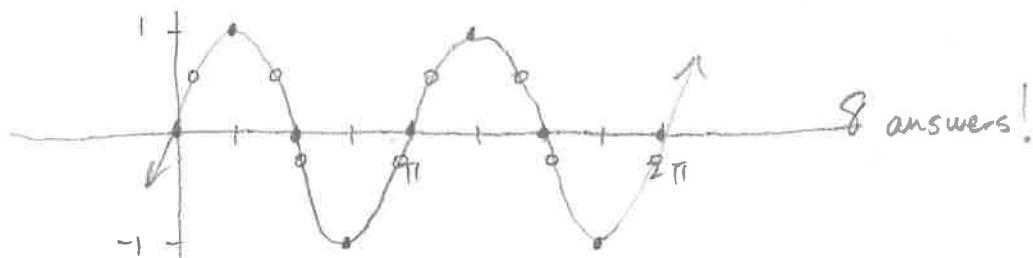
$$\text{also } 3.481 + 2\pi \\ 5.943 + 2\pi$$

$$2x = 3.481, 5.943, 9.764, 12.226$$

$$x = 1.741, 2.972, 4.883, 6.113$$

$$ii) x = \frac{\pi}{12} + n\pi, \frac{5\pi}{12} + n\pi, 1.741 + n\pi, 2.972 + n\pi$$

$n \in \text{INTEGERS}$



$$l) \sin^2 2x - \sin 2x - 2 = 0$$

$$i) (\sin 2x - 2)(\sin 2x + 1) = 0$$

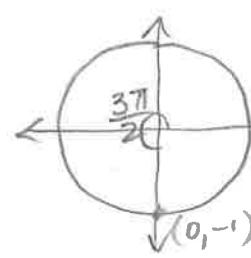
$$\sin 2x = 2$$

reject

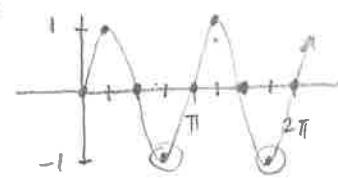
$$(-1 \leq \sin 2x \leq 1)$$

$$\sin 2x = -1$$

QUADRANTAL



$$\sin x = y\text{-value}$$



2 answers

$$2x = \sin^{-1}(-1)$$

$$2x = \frac{3\pi}{2} \quad \text{and} \quad 2x = \frac{3\pi}{2} + 2\pi$$

$$x = \frac{3\pi}{4}, \frac{7\pi}{4}$$

$$ii) x = \frac{3\pi}{4} + n\pi \quad (n \text{ an integer})$$

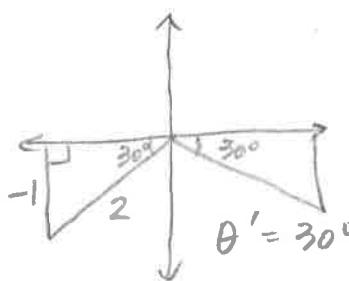
q2: Solve  $2\sin 3\theta + 1 = 0$  i)  $0^\circ < \theta < 360^\circ$  ii) General form

$$i) 2\sin 3\theta + 1 = 0$$

$$2\sin 3\theta = -1$$

$$\sin 3\theta = -\frac{1}{2}$$

$$3\theta = \sin^{-1}\left(-\frac{1}{2}\right)$$



Since dividing by 3 later

$$3\theta = 210^\circ, 330^\circ, 210^\circ + 360^\circ, 330^\circ + 360^\circ, 210^\circ + 720^\circ, 330^\circ + 720^\circ$$

$$3\theta = 210^\circ, 330^\circ, 570^\circ, 690^\circ, 930^\circ, 1050^\circ$$

$$\theta = 70^\circ, 110^\circ, 190^\circ, 230^\circ, 310^\circ, 350^\circ$$

$$ii) \theta = 70^\circ + n120^\circ, 110^\circ + n120^\circ \quad n \in \text{INTEGERS}$$

## Ch. 7.1 - Trigonometric Identities and Equations

- an EQUATION is true for some values of the variable  
(e.g.:  $\sin x - 1 = 0$ )

$$\sin x = 1 \quad (\text{recall: } \sin x = \frac{y\text{-value}}{r} = \frac{y}{r})$$

$$x = \frac{\pi}{2} \text{ and all values coterminal}.$$

- an IDENTITY is true for all values of the variable

$$\text{e.g.: } 2x = 3x - x$$

$$2x = 2x$$

$$1 = 1 \quad \text{OR} \quad 0 = 0 \quad \text{ALWAYS true!}$$

- there exist infinitely many trig. identities, however, we will focus on a number of BASIC identities.

Recall: If  $\theta$  is an angle in standard position with  $P(x, y)$  on  $\theta$ 's terminal side, then the six trig. ratios are:

$$\sin \theta = \frac{y}{r}$$

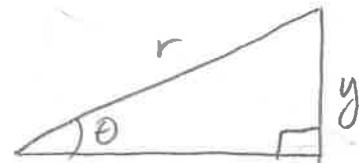
$$\csc \theta = \frac{r}{y}$$

$$\cos \theta = \frac{x}{r}$$

$$\sec \theta = \frac{r}{x}$$

$$\tan \theta = \frac{y}{x}$$

$$\cot \theta = \frac{x}{y}$$



$$r^2 = x^2 + y^2$$

$$r = \sqrt{x^2 + y^2}$$

Notice, then, that:

$$\sin \theta \cdot \csc \theta = \frac{y}{r} \cdot \frac{r}{y} = 1$$

$$\cos \theta \cdot \sec \theta = \frac{x}{r} \cdot \frac{r}{x} = 1$$

$$\tan \theta \cdot \cot \theta = \frac{y}{x} \cdot \frac{x}{y} = 1$$

all of these ratios are RECIPROCALS of each other!

## The Reciprocal Identities

$$1. \csc \theta = \boxed{\frac{1}{\sin \theta}}$$

$$2. \sec \theta = \boxed{\frac{1}{\cos \theta}}$$

$$3. \cot \theta = \boxed{\frac{1}{\tan \theta}}$$

\* these are all IDENTITIES because they are always true for all allowable (within domain) values of the variable.

Also, notice:

$$\frac{\sin \theta}{\cos \theta} = \frac{\frac{y}{r}}{\frac{x}{r}} = \frac{y}{r} \cdot \frac{r}{x} = \frac{y}{x} = \tan \theta$$

$$\frac{\cos \theta}{\sin \theta} = \frac{\frac{x}{r}}{\frac{y}{r}} = \frac{x}{r} \cdot \frac{r}{y} = \frac{x}{y} = \cot \theta$$

## The Quotient Identities

$$1. \tan \theta = \boxed{\frac{\sin \theta}{\cos \theta}}$$

$$2. \cot \theta = \boxed{\frac{\cos \theta}{\sin \theta}}$$

Next: examine  $\sin^2 \theta + \cos^2 \theta$

NOTE:  $\sin^2 \theta = \underline{(\sin \theta)^2}$

i)  $\sin^2 \theta + \cos^2 \theta = \left( \frac{y}{r} \right)^2 + \left( \frac{x}{r} \right)^2 = \frac{y^2}{r^2} + \frac{x^2}{r^2}$

$= \frac{y^2 + x^2}{r^2} = \frac{r^2}{r^2} = 1$

\*   
 $x^2 + y^2 = r^2$

ii) start with  $\sin^2 \theta + \cos^2 \theta = 1$ , and divide each term by  $\cos^2 \theta$ :

$$\frac{\sin^2 \theta}{\cos^2 \theta} + \frac{\cos^2 \theta}{\cos^2 \theta} = \frac{1}{\cos^2 \theta}$$

$$\frac{(\sin \theta)^2}{(\cos \theta)^2} + 1 = \frac{1}{(\cos \theta)^2}$$

$$\left( \frac{\sin \theta}{\cos \theta} \right)^2 + 1 = \left( \frac{1}{\cos \theta} \right)^2$$

$$\tan^2 \theta + 1 = \sec^2 \theta$$

iii) start with  $\sin^2 \theta + \cos^2 \theta = 1$ , and divide each term by  $\sin^2 \theta$ :

$$\frac{\sin^2 \theta}{\sin^2 \theta} + \frac{\cos^2 \theta}{\sin^2 \theta} = \frac{1}{\sin^2 \theta}$$

$$1 + \left(\frac{\cos \theta}{\sin \theta}\right)^2 = \left(\frac{1}{\sin \theta}\right)^2$$

$$1 + \cot^2 \theta = \csc^2 \theta$$

### The Pythagorean Identities

$$1. \sin^2 \theta + \cos^2 \theta = 1$$

$$2. 1 + \tan^2 \theta = \sec^2 \theta$$

$$3. 1 + \cot^2 \theta = \csc^2 \theta$$

### Summary: Fundamental Trig. Identities:

$$\textcircled{1} \quad \csc \theta = \frac{1}{\sin \theta}$$

$$\textcircled{2} \quad \sec \theta = \frac{1}{\cos \theta}$$

$$\textcircled{3} \quad \cot \theta = \frac{1}{\tan \theta}$$

$$\textcircled{4} \quad \tan \theta = \frac{\sin \theta}{\cos \theta}$$

$$\textcircled{5} \quad \cot \theta = \frac{\cos \theta}{\sin \theta}$$

$$\textcircled{6} \quad \sin^2 \theta + \cos^2 \theta = 1$$

$$\textcircled{7} \quad 1 + \tan^2 \theta = \sec^2 \theta \quad \textcircled{8} \quad 1 + \cot^2 \theta = \csc^2 \theta$$

## Strategies for Simplifying a Trig. Expression

1) Find and convert to a common denominator:

$$\text{eg: } \sin x + \frac{\sin x}{\cos x}$$

$$\begin{aligned} &= \frac{\sin x \cos x}{\cos x} + \frac{\sin x}{\cos x} \\ &= \frac{\sin x \cos x + \sin x}{\cos x} \end{aligned}$$

2) Factor:

$$\text{eg: } \frac{\sin x \cos x + \sin x}{\cos x}$$

$$= \frac{\sin x (\cos x + 1)}{\cos x}$$

$$= \tan x (\cos x + 1)$$

$$\text{eg: } 1 - \sin^2 x$$

$$= (1 + \sin x)(1 - \sin x)$$

3) Change all terms to sine and cosine:

$$\text{eg: } \frac{\sin x}{\csc x} + \frac{\cos x}{\sec x}$$

$$\text{eg: } \frac{\tan x}{\sec x}$$

$$= \frac{\sin x}{\left(\frac{1}{\sin x}\right)} + \frac{\cos x}{\left(\frac{1}{\cos x}\right)}$$

$$= \frac{\sin x}{\cos x} \cdot \frac{1}{\frac{1}{\cos x}}$$

$$= \sin^2 x + \cos^2 x$$

$$= \frac{\sin x \cos x}{\cos x}$$

$$= \sin x$$

4) Conjugate (multiply by the complement):

eg:  $\frac{1}{1-\cos x}$

$$\begin{aligned} &= \frac{1}{1-\cos x} \cdot \frac{1+\cos x}{1+\cos x} \\ &= \frac{1+\cos x}{1-\cos^2 x} \\ &= \frac{1+\cos x}{\sin^2 x} \end{aligned}$$

\* recall:

$$\frac{\sqrt{2}}{\sqrt{5}-1}$$

$$\boxed{\frac{\sqrt{5}+1}{\sqrt{5}-1}}$$

eg: Simplify the following:

a)  $(\sec^2 x - 1) \cot^2 x$

$$= ((1 + \tan^2 x) - 1) \cot^2 x$$

$$= (\tan^2 x) (\cot^2 x)$$

$$= (\tan^2 x) \left( \frac{1}{\tan^2 x} \right)$$

$$= \boxed{1}$$

b)  $\frac{2 \cos x}{1 - \sin^2 x}$

$$= \frac{2 \cos x}{\cos^2 x}$$

$$= \frac{2 \cos x}{\cos x \cdot \cos x}$$

$$= \frac{2}{\cos x}$$

$$= \boxed{2 \sec x}$$

eg2: Simplify the following:

$$a) \frac{\sin x}{1+\cos x} + \frac{\sin x}{1-\cos x}$$

$$= \frac{\sin x}{1+\cos x} \cdot \frac{1-\cos x}{1-\cos x} + \frac{\sin x}{1-\cos x} \cdot \frac{1+\cos x}{1+\cos x}$$

$$= \frac{(\sin x)(1-\cos x)}{1-\cos^2 x} + \frac{(\sin x)(1+\cos x)}{1-\cos^2 x}$$

$$= \frac{(\sin x)(1-\cos x) + (\sin x)(1+\cos x)}{\sin^2 x}$$

$$= \frac{\sin x (1-\cos x + 1+\cos x)}{\sin^2 x}$$

$$= \frac{2}{\sin x} = \boxed{2 \csc x}$$

over →

$$\begin{aligned}
 b) \quad & \frac{1 + \sin x}{\cos x} - \frac{\cos x}{1 - \sin x} \\
 = & \left( \frac{1 + \sin x}{\cos x} \right) \left( \frac{1 - \sin x}{1 - \sin x} \right) - \left( \frac{\cos x}{1 - \sin x} \right) \left( \frac{\cos x}{\cos x} \right) \\
 = & \frac{(1 - \sin^2 x) - \cos^2 x}{\cos x (1 - \sin x)} \\
 = & \frac{\cos^2 x - \cos^2 x}{\cos x (1 - \sin x)} \\
 = & \boxed{0}
 \end{aligned}$$

eg3: Simplify the following:

$$a) \frac{\sin x \cos x + \sin x}{\cos x + \cos^2 x}$$

$$= \frac{\sin x (\cos x + 1)}{\cos x (1 + \cos x)}$$

$$= \frac{\sin x}{\cos x} = \boxed{\tan x}$$

$$b) \frac{\cos x \cot x + \cos x}{\cot x + \cot^2 x}$$

$$= \frac{\cos x (\cot x + 1)}{\cot x (1 + \cot x)}$$

$$= \frac{\cos x}{\left( \frac{\cos x}{\sin x} \right)} = \boxed{\sin x}$$

Eg 4: Determine the restriction(s) upon  $x$  in the expression  $\tan x + \csc x$  ( $0 \leq x < 2\pi$ ):

$$= \frac{\sin x}{\cos x} + \frac{1}{\sin x}$$

$$\text{Solve: } \cos x = 0$$

$$x = \frac{\pi}{2}, \frac{3\pi}{2}$$

$$\sin x = 0$$

$$x = 0, \pi$$

$$\cos x = \frac{x\text{-value}}{1}$$

$$\sin x = \frac{y\text{-value}}{1}$$

$$\therefore x \neq 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}.$$

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# 1-9

## Ch. 7.2 - Verifying Trig Identities

- the key to proving/verifying identities is to utilize the eight fundamental identities along with basic algebra rules to rewrite trig expressions.

Def'ns:

EXPRESSION - has no equal sign. It is merely the sum/difference/product/quotient of functions.

EQUATION - a statement that is true for a set of specific values (ie. a conditional statement).

IDENTITY - an equation that is true for all real values.

- NOTE: proving an identity is quite different from solving an equation.

Helpful Rules for Proving/Verifying Identities:

- ↓  
no particular order
- there may be > 1 way to proving identities
- ① Change all trig values to sine and/or cosine;
  - ② Write an expression with a common denominator;
  - ③ Remember the conjugate step and how to factor;
  - ④ Work with one side of an equation at a time, starting with the most 'complicated' side.

\* Proven when left <sup>column</sup> statement = right <sup>column</sup> statement

e.g.: Prove the identity:  $\frac{\csc^2 \theta - 1}{\csc^2 \theta} = \cos^2 \theta$

$$\frac{\csc^2 \theta - 1}{\csc^2 \theta} = \cos^2 \theta$$

$$\frac{\cot^2 \theta}{\csc^2 \theta} =$$

$$\frac{\frac{\cos^2 \theta}{\sin^2 \theta}}{\frac{1}{\sin^2 \theta}} =$$

$$\frac{\cos^2 \theta \sin^2 \theta}{\sin^2 \theta} =$$

$$\boxed{\cos^2 \theta = \cos^2 \theta}$$

OR

$$\frac{\csc^2 \theta - 1}{\csc^2 \theta} = \cos^2 \theta$$

$$\frac{\csc^2 \theta}{\csc^2 \theta} - \frac{1}{\csc^2 \theta}$$

$$1 - \sin^2 \theta$$

$$\boxed{\cos^2 \theta = \cos^2 \theta}$$

Q2: Prove the identity:  $\frac{1}{1-\cos\alpha} + \frac{1}{1+\cos\alpha} = 2\csc^2\alpha$

$$\frac{1}{1-\cos\alpha} + \frac{1}{1+\cos\alpha} = 2\csc^2\alpha$$

$$\left(\frac{1}{1-\cos\alpha}\right)\left(\frac{1+\cos\alpha}{1+\cos\alpha}\right) + \left(\frac{1}{1+\cos\alpha}\right)\left(\frac{1-\cos\alpha}{1-\cos\alpha}\right) = \frac{2}{\sin^2\alpha}$$

$$\frac{1+\cos\alpha + 1-\cos\alpha}{1-\cos^2\alpha} = \frac{2}{\sin^2\alpha}$$

$$\frac{2}{1-\cos^2\alpha} =$$

$$\boxed{\frac{2}{\sin^2\alpha}} = \frac{2}{\sin^2\alpha}$$

Q3: Prove the identity  $\tan x + \cot x = \sec x \csc x$

$$\tan x + \cot x = \sec x \csc x$$

$$\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} = \frac{1}{\cos x \sin x}$$

$$\left(\frac{\sin x}{\cos x}\right)\left(\frac{\sin x}{\cos x}\right) + \left(\frac{\cos x}{\sin x}\right)\left(\frac{\cos x}{\sin x}\right) =$$

$$\frac{\sin^2 x + \cos^2 x}{\cos x \sin x} =$$

$$\boxed{\frac{1}{\cos x \sin x}} = \frac{1}{\cos x \sin x}$$

eg4: Prove the identity:  $\csc x + \cot x = \frac{\sin x}{1 - \cos x}$

$$\csc x + \cot x = \frac{\sin x}{1 - \cos x}$$

$$\frac{1}{\sin x} + \frac{\cos x}{\sin x} = \left( \frac{\sin x}{1 - \cos x} \right) \left( \frac{1 + \cos x}{1 + \cos x} \right)$$

$$\frac{\cos x + 1}{\sin x} = \frac{\sin x + \sin x \cos x}{1 - \cos^2 x} \quad \begin{matrix} \text{don't} \\ \text{have to} \\ \text{expand} \end{matrix}$$

$$= \frac{\sin x (1 + \cos x)}{\sin^2 x}$$

$$\boxed{\frac{\cos x + 1}{\sin x} = \frac{1 + \cos x}{\sin x}}$$

eg5: Prove the identity:  $\frac{\tan^2 x}{1 + \sec x} = \frac{1 - \cos x}{\cos x}$

$$\frac{\tan^2 x}{1 + \sec x} = \frac{1 - \cos x}{\cos x}$$

$$\frac{\sec^2 x - 1}{1 + \sec x} =$$

$$\frac{(\sec x + 1)(\sec x - 1)}{1 + \sec x}$$

$$\sec x - 1 =$$

$$\frac{1}{\cos x} - 1 =$$

$$\frac{1 - \cos x}{\cos x} = \frac{1 - \cos x}{\cos x}$$

p. 311  
# 1-26

typo: # 23 p. 314

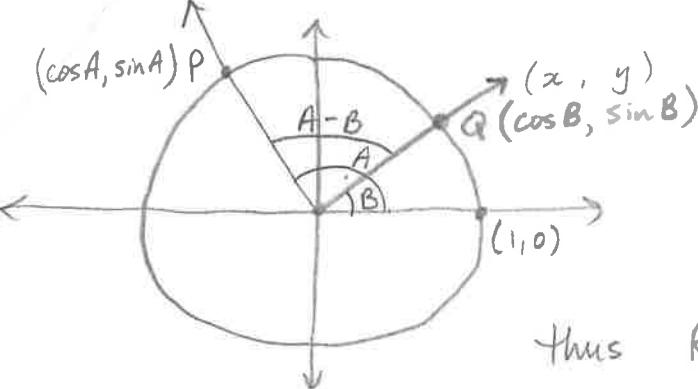
23)  $\tan x (\csc x + 1)$

## Ch. 7.4 - Sum and Difference Identities

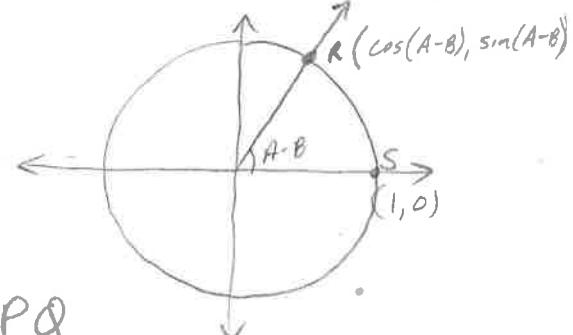
Start with:  $\cos(A - B) = \cos A \cos B + \sin A \sin B$

Proof:

Unit Circle



Re-draw with angle  $(A - B)$  in standard position



$$\text{thus } RS = PQ$$

$$\therefore (RS)^2 = (PQ)^2$$

\* Recall: distance formula:  $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$$(RS)^2 = (PQ)^2$$

$$\sqrt{(\cos(A-B) - 1)^2 + (\sin(A-B) - 0)^2} = \sqrt{(\cos A - \cos B)^2 + (\sin A - \sin B)^2}$$

$$(\cos(A-B) - 1)^2 + (\sin(A-B))^2 = (\cos A - \cos B)^2 + (\sin A - \sin B)^2$$

$$\cos^2(A-B) - 2\cos(A-B) + 1 + \sin^2(A-B) = \cos^2 A - 2\cos A \cos B + \cos^2 B + \sin^2 A - 2\sin A \sin B + \sin^2 B$$

$$(\sin^2(A-B) + \cos^2(A-B)) - 2\cos(A-B) + 1 = (\sin^2 A + \cos^2 A) + (\sin^2 B + \cos^2 B)$$

$$- 2\cos A \cos B - 2\sin A \sin B$$

$$1 + 1 - 2\cos(A-B) = 1 + 1 - 2\cos A \cos B - 2\sin A \sin B$$

$$2 - 2\cos(A-B) = 2 - 2\cos A \cos B - 2\sin A \sin B$$

$$- 2\cos(A-B) = - 2(\cos A \cos B + \sin A \sin B)$$

$$\boxed{\cos(A - B) = \cos A \cos B + \sin A \sin B}$$

Recall:  $\cos(-x) = \underline{\cos x}$  and  $\sin(-x) = \underline{-\sin x}$

so, if  $\cos(A-B) = \cos A \cos B + \sin A \sin B$

then,  $\cos(A-(-B)) = \cos A \cos(-B) + \sin A \sin(-B)$

$$\boxed{\cos(A+B) = \cos A \cos B - \sin A \sin B}$$

Recall:  $\sin x = \underline{\cos(\frac{\pi}{2} - x)}$  and  $\cos x = \underline{\sin(\frac{\pi}{2} - x)}$

$$\begin{aligned} \text{so, } \sin(A+B) &= \cos(\frac{\pi}{2} - (A+B)) \\ &= \cos((\frac{\pi}{2} - A) - B) \\ &= \cos(\frac{\pi}{2} - A) \cos B + \sin(\frac{\pi}{2} - A) \sin B \end{aligned}$$

$$\boxed{\sin(A+B) = \sin A \cos B + \cos A \sin B}$$

$$\therefore \sin(A+(-B)) = \sin A \cos(-B) + \cos A \sin(-B)$$

$$\boxed{\sin(A-B) = \sin A \cos B - \cos A \sin B}$$

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$$

\* divide each term by  $\cos A \cos B$

$$= \frac{\frac{\sin A \cos B}{\cos A \cos B}}{\frac{\cos A \cos B}{\cos A \cos B}} + \frac{\frac{\cos A \sin B}{\cos A \cos B}}{\frac{\cos A \cos B}{\cos A \cos B}}$$
$$= \frac{\sin A}{\cos A} - \frac{\sin B}{\cos B}$$

$$= \left( \frac{\sin A}{\cos A} \right) + \left( \frac{\sin B}{\cos B} \right)$$

$$- \left( \frac{\sin A}{\cos A} \right) \left( \frac{\sin B}{\cos B} \right)$$

$$\boxed{\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}}$$

again, if  $\sin(-x) = -\sin x$  and  $\cos(-x) = \cos x$ ,

then:

$$\tan(-x) = \frac{\sin(-x)}{\cos(-x)} = \frac{-\sin x}{\cos x} = -\tan x$$

thus,  $\tan(A + (-B)) = \frac{\tan A + \tan(-B)}{1 - \tan A \tan(-B)}$

$$\boxed{\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}}$$

Sum and Difference Identities Summarized:

$$\sin(A+B) = \underline{\sin A \cos B + \cos A \sin B}$$

$$\sin(A-B) = \underline{\sin A \cos B - \cos A \sin B}$$

$$\cos(A+B) = \underline{\cos A \cos B - \sin A \sin B}$$

$$\cos(A-B) = \underline{\cos A \cos B + \sin A \sin B}$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

# Even-Odd & Cofunction Identities Summarized:

$$\sin(-A) = \underline{-\sin A} \quad \cos(-A) = \underline{\cos A} \quad \tan(-A) = \underline{-\tan A}$$

$$\sin\left(\frac{\pi}{2}-A\right) = \underline{\cos A} \quad \cos\left(\frac{\pi}{2}-A\right) = \underline{\sin A}$$

$$\csc\left(\frac{\pi}{2}-A\right) = \underline{\sec A} \quad \sec\left(\frac{\pi}{2}-A\right) = \underline{\csc A}$$

$$\tan\left(\frac{\pi}{2}-A\right) = \underline{\cot A} \quad \Rightarrow \quad \boxed{\frac{\sin\left(\frac{\pi}{2}-A\right)}{\cos\left(\frac{\pi}{2}-A\right)} = \frac{\cos A}{\sin A}}$$

$$\cot\left(\frac{\pi}{2}-A\right) = \underline{\tan A}$$

- the sum and difference identities, along with the even-odd and cofunction identities, can be used to solve a wide variety of problems in trig:

e.g.: Find the exact value of  $\cos 105^\circ$ .

Use special angles:

$$\cos 105^\circ = \cos(60^\circ + 45^\circ)$$

$$= \cos 60^\circ \cos 45^\circ - \sin 60^\circ \sin 45^\circ$$

$$= \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right) - \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{2\sqrt{2}} - \frac{\sqrt{3}}{2\sqrt{2}}$$

$$= \frac{1-\sqrt{3}}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \boxed{\frac{\sqrt{2}-\sqrt{6}}{4}}$$

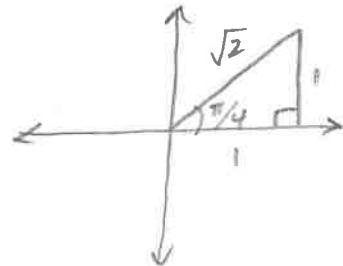
eg2: Simplify  $\frac{\tan \frac{2\pi}{5} - \tan \frac{3\pi}{20}}{1 + \tan \frac{2\pi}{5} \tan \frac{3\pi}{20}}$

$$= \tan \left( \frac{2\pi}{5} - \frac{3\pi}{20} \right)$$

$$= \tan \left( \frac{8\pi - 3\pi}{20} \right)$$

$$= \tan \left( \frac{\pi}{4} \right)$$

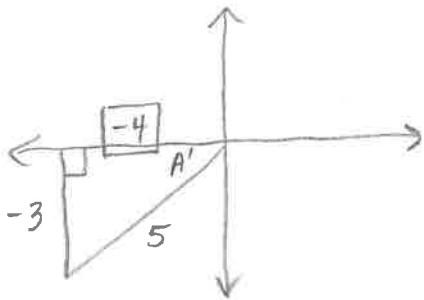
$$= \boxed{1}$$



eg3: Given  $\sin A = -\frac{3}{5}$ , with  $A$  in  $\text{QIII}$ , and  $\cos B = \frac{5}{13}$ , with  $B$  in  $\text{QIV}$ , find  $\sin(A+B)$ .

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

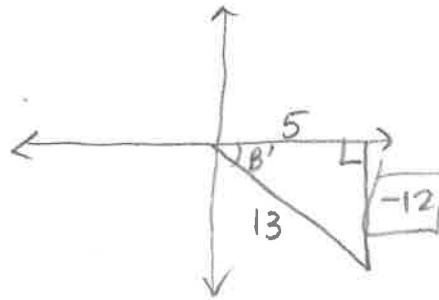
↑      ↑  
\* need



$$a^2 + b^2 = c^2$$

$$(-3)^2 + b^2 = 5^2$$

$$b = \pm 4 = \boxed{-4}$$



$$a^2 + b^2 = c^2$$

$$5^2 + b^2 = 13^2$$

$$b = \pm 12  
= \boxed{-12}$$

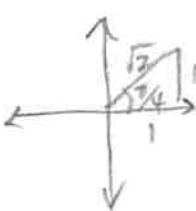
$$\sin(A+B) = \left(-\frac{3}{5}\right)\left(\frac{5}{13}\right) + \left(\frac{-4}{5}\right)\left(\frac{-12}{13}\right)$$

$$= -\frac{15}{65} + \frac{48}{65} = \boxed{\frac{33}{65}}$$

Eg4: Solve:  $\sin(x + \frac{\pi}{4}) + \sin(x - \frac{\pi}{4}) = -1$   
 $(0 \leq x < 2\pi)$

$$\left[ \sin x \cos \frac{\pi}{4} + \cos x \sin \frac{\pi}{4} \right] + \left[ \sin x \cos \frac{\pi}{4} - \cos x \sin \frac{\pi}{4} \right] = -1$$

$$\left[ (\sin x) \left( \frac{1}{\sqrt{2}} \right) + (\cos x) \left( \frac{1}{\sqrt{2}} \right) \right] + \left[ (\sin x) \left( \frac{1}{\sqrt{2}} \right) - (\cos x) \left( \frac{1}{\sqrt{2}} \right) \right] = -1$$

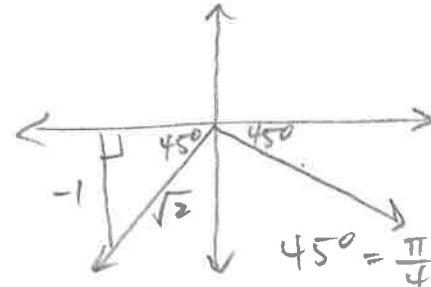


$$\frac{2}{\sqrt{2}} \sin x = -1$$

$$\sin x = -\frac{\sqrt{2}}{2} \quad \text{irrationalize}$$

$$\sin x = -\frac{1}{\sqrt{2}}$$

$$x = \frac{5\pi}{4}, \frac{7\pi}{4}$$



Eg5: Prove the identity:

$$\sin(A+B) + \sin(A-B) = 2 \sin A \cos B$$

$$\sin A \cos B + \cos A \sin B + \sin A \cos B - \cos A \sin B$$

$$= 2 \sin A \cos B$$

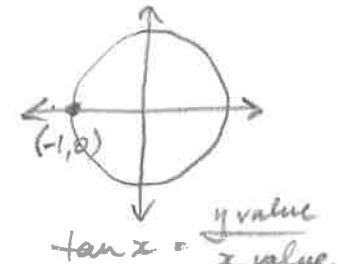
$$2 \sin A \cos B = 2 \sin A \cos B$$

\* this is eg 4!

eg6: Find the general form of the solution to

$$2 \tan x + \tan(\pi - x) = \sqrt{3}$$

$$2 \tan x + \left( \frac{\tan \pi - \tan x}{1 + \tan \pi \tan x} \right) = \sqrt{3}$$



$$\tan \pi = \frac{0}{-1} = 0$$

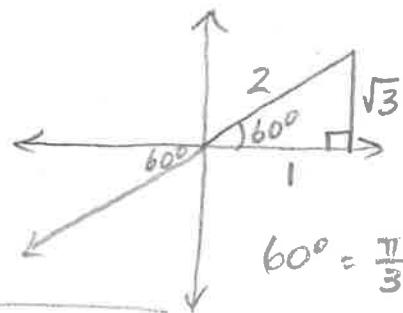
$$2 \tan x + \left( \frac{0 - \tan x}{1 + 0(\tan x)} \right) = \sqrt{3}$$

$$2 \tan x + \left( \frac{-\tan x}{1} \right) = \sqrt{3}$$

$$\tan x = \sqrt{3}$$

$$x = \tan^{-1} \sqrt{3}$$

$$x = \frac{\pi}{3}, \frac{4\pi}{3}$$



$$60^\circ = \frac{\pi}{3}$$

$$x = \frac{\pi}{3} + n\pi \quad (n \in \text{INTEGERS})$$

eg 7: Determine the amplitude, period, and phase shift of  $f(x) = 3\sqrt{2} \sin 2x \cos \frac{\pi}{4} + 3\sqrt{2} \cos 2x \sin \frac{\pi}{4}$

$$\begin{aligned}f(x) &= 3\sqrt{2} \left( \sin 2x \cos \frac{\pi}{4} + \cos 2x \sin \frac{\pi}{4} \right) \\&= 3\sqrt{2} \left( \sin \left( 2x + \frac{\pi}{4} \right) \right) \\&= 3\sqrt{2} \sin 2 \left( x + \frac{\pi}{8} \right)\end{aligned}$$

Amplitude =  $|3\sqrt{2}| = \boxed{3\sqrt{2}}$

Period =  $\frac{2\pi}{2} = \boxed{\pi}$

Phase Shift:  $x + \frac{\pi}{8} = 0$  to  $x + \frac{\pi}{8} = \pi$   
 $x = -\frac{\pi}{8}$                                $x = \pi - \frac{\pi}{8}$

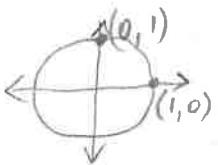
$\frac{\pi}{8}$  left or  $\boxed{-\frac{\pi}{8}}$

eg8: Simplify:

$$\csc(90^\circ - \theta) \sec(360^\circ - \theta) - \tan(720^\circ + \theta) \cot(450^\circ - \theta)$$

$$= \left( \frac{1}{\sin(90^\circ - \theta)} \right) \left( \frac{1}{\cos(360^\circ - \theta)} \right) - \left[ \tan 720^\circ + \theta \left( \frac{\cos(450^\circ - \theta)}{\sin(450^\circ - \theta)} \right) \right]$$

$$= \left( \frac{1}{\cos \theta} \right) \left( \frac{1}{\cos 360^\circ \cos \theta + \sin 360^\circ \sin \theta} \right) - \left[ \left( \frac{\tan 720^\circ + \tan \theta}{1 - \tan 720^\circ \tan \theta} \right) \right] \quad \text{below}$$



$$\left( \frac{\cos 450^\circ \cos \theta + \sin 450^\circ \sin \theta}{\sin 450^\circ \cos \theta - \cos 450^\circ \sin \theta} \right)$$

$$= \left( \frac{1}{\cos \theta} \right) \left( \frac{1}{(1) \cos \theta + (0) \sin \theta} \right) - \left[ \left( \frac{0 + \tan \theta}{1 - 0 \tan \theta} \right) \left( \frac{0 \cos \theta + 1 \sin \theta}{1 \cos \theta - 0 \sin \theta} \right) \right]$$

$$= \left( \frac{1}{\cos \theta} \right) \left( \frac{1}{\cos \theta} \right) - \left[ \tan \theta \left( \frac{\sin \theta}{\cos \theta} \right) \right]$$

$$= \frac{1}{\cos^2 \theta} - \left[ \left( \frac{\sin \theta}{\cos \theta} \right) \left( \frac{\sin \theta}{\cos \theta} \right) \right]$$

$$= \frac{1 - \sin^2 \theta}{\cos^2 \theta} = \frac{\cos^2 \theta}{\cos^2 \theta} = \boxed{1}$$

p. 332

# 1-9

omit 5b

$$\begin{aligned}& \csc(90^\circ - \theta) \sec(360^\circ - \theta) - \tan(720^\circ + \theta) \cot(450^\circ - \theta) \\&= \sec \theta \sec(-\theta) - \tan \theta \cot(90^\circ - \theta) \\&= \sec \theta \left( \frac{1}{\cos(-\theta)} \right) - \tan \theta \tan \theta \\&= \sec \theta \sec \theta - \tan \theta \tan \theta \\&= 1\end{aligned}$$

## Ch. 7.5 - Double-Angle Identities

- the sum and difference identities from Ch. 6.4 may be used to generate even more trig. identities.

Start with:

$$\sin 2A = 2 \sin A \cos A$$

the double-angle formula for sine

Derivation:  $\sin(A+B) = \sin A \cos B + \cos A \sin B$

\* replace B

$$\text{with } A : \sin(A+A) = \sin A \cos A + \cos A \sin A$$

$$\boxed{\sin 2A = 2 \sin A \cos A}$$

Similarly:  $\cos(A+B) = \cos A \cos B - \sin A \sin B$

\* replace B

$$\text{with } A : \cos(A+A) = \cos A \cos A - \sin A \sin A$$

$$\boxed{\cos 2A = \cos^2 A - \sin^2 A}$$

$$\text{recall: } \sin^2 A = 1 - \cos^2 A$$

$$\text{so, } \cos 2A = \cos^2 A - (1 - \cos^2 A)$$

$$\boxed{\cos 2A = 2 \cos^2 A - 1}$$

$$\text{or, recall: } \cos^2 A = 1 - \sin^2 A$$

$$\cos 2A = 2(1 - \sin^2 A) - 1$$

$$= 2 - 2 \sin^2 A - 1$$

$$\boxed{\cos 2A = 1 - 2 \sin^2 A}$$

} three different formulae

Again, similarly:

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

\* replace  $B$  with  $A$ :  $\tan(A+A) = \frac{\tan A + \tan A}{1 - \tan A \tan A}$

$$\boxed{\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}}$$

### Summary: Double-Angle Identities

$$\sin 2A = 2 \sin A \cos A$$

$$\begin{aligned}\cos 2A &= \cos^2 A - \sin^2 A \\ &= 2 \cos^2 A - 1\end{aligned}$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

$$= 1 - 2 \sin^2 A$$

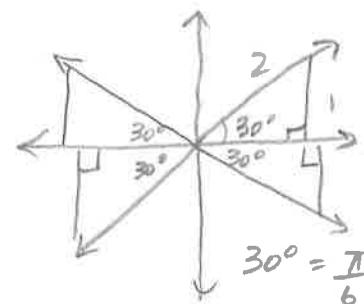
e.g. Solve  $\cos 2x = 2 \sin^2 x$  ( $0 \leq x < 2\pi$ )

$$1 - 2 \sin^2 x = 2 \sin^2 x$$

$$1 = 4 \sin^2 x$$

$$\sqrt{\sin^2 x} = \sqrt{\frac{1}{4}}$$

$$\sin x = \pm \frac{1}{2}$$



$$x = \sin^{-1}\left(\frac{1}{2}\right) \quad x = \sin^{-1}\left(-\frac{1}{2}\right)$$

$$\boxed{x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}}$$

eg2: Use the double-angle formulas to simplify:

a)  $12 \sin 4x \cos 4x$

$$= 6(2 \sin 4x \cos 4x)$$

$$= 6(\sin 2(4x))$$

$$\boxed{= 6(\sin 8x)}$$

b)  $4 - 8 \cos^2 6x$

$$= -4(-1 + 2 \cos^2 6x)$$

$$= -4(2 \cos^2 6x - 1)$$

$$= -4(\cos 2(6x))$$

$$\boxed{= -4(\cos 12x)}$$

c)  $\frac{4 \tan 3x}{1 - \tan^2 3x}$

$$= 2 \frac{(2 \tan 3x)}{1 - \tan^2 3x}$$

$$= 2(\tan 2(3x))$$

$$\boxed{= 2 \tan 6x}$$

Eg3: Prove the identity:  $\frac{\sin 6x}{1 + \cos 6x} = \tan 3x$

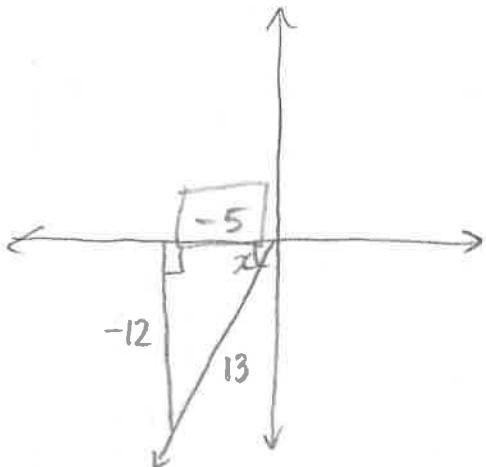
$$\frac{\sin 6x}{1 + \cos 6x} = \tan 3x$$

$$\frac{2 \sin 3x \cos 3x}{1 + (2 \cos^2 3x - 1)} = \frac{\sin 3x}{\cos 3x}$$

$$\frac{2 \sin 3x \cos 3x}{2 \cos^2 3x}$$

$$\boxed{\frac{\sin 3x}{\cos 3x} = \frac{\sin 3x}{\cos 3x}}$$

Eg4: Given  $\sin x = -\frac{12}{13}$  in  $\text{QIII}$ , find  $\tan 2x$ .



$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$$

$$= \frac{2 \left( \frac{12}{5} \right)}{1 - \left( \frac{12}{5} \right)^2}$$

$$= \frac{\frac{24}{5}}{1 - \frac{144}{25}}$$

$$= \frac{\frac{24}{5}}{-\frac{119}{25}}$$

$$= \frac{\frac{24}{5}}{-\frac{119}{25}} = \frac{(24)(25)}{(-119)} = \boxed{\frac{-120}{119}}$$

$$a^2 + b^2 = c^2$$

$$(-12)^2 + b^2 = 13^2$$

$$b = \pm 5 = \boxed{-5}$$

## Power - Reducing Identities

- these are the double-angle identities written in a different way:

$$\cos 2x = 1 - 2\sin^2 x \quad (\text{solve for } \sin^2 x)$$

$$2\sin^2 x = 1 - \cos 2x$$

$$\boxed{\sin^2 x = \frac{1 - \cos 2x}{2}}$$

Power - Reducing

2nd degree

1st degree

$$\cos 2x = 2\cos^2 x - 1 \quad (\text{solve for } \cos^2 x)$$

$$2\cos^2 x = \cos 2x + 1$$

$$\boxed{\cos^2 x = \frac{\cos 2x + 1}{2}}$$

$$\tan^2 x = \frac{\sin^2 x}{\cos^2 x} = \frac{\frac{1 - \cos 2x}{2}}{\frac{1 + \cos 2x}{2}} = \frac{1 - \cos 2x}{1 + \cos 2x}$$

### Summary:

$$\sin^2 A = \frac{1 - \cos 2A}{2} = \frac{1}{2} (1 - \cos 2A)$$

$$\cos^2 A = \frac{1 + \cos 2A}{2} = \frac{1}{2} (1 + \cos 2A)$$

$$\tan^2 A = \frac{1 - \cos 2A}{1 + \cos 2A}$$

\*typo: p 343 #5b  
 $\cos 2x$

p 340  
#1-15  
9, 10, 11 for fun!

+ Ch Rev. OMIT

#15, 16 Graph calc.